J. Fourier, 1808-9: "Regarding the researches of d'Alembert and Euler could one not add that if they know this expansion, they made but a very imperfect use of it. They were both persuaded that an arbitrary and discontinuous function could never be resolved in series of this kind, and it does not even seem that anyone had developed a constant in cosines of mulitiple arcs, the first probelm which I had to solve in the theory of heat."

FOURIER ANALYSIS

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These notes (written from 2018-10-24) for TMA4170 are based on Dym–Mckean's monograph [3], Stein–Shakarchi's book [12] and Boggess–Narcowich's book [1].

The purpose of the notes is to give a mathematical account of Fourier ideas on the circle and the line. The emphasis is placed on the applications, which include

1). Eigenfunction expansion for the Dirac operator $-i\frac{d}{dx}$;
2). Wirtinger and Poincaré inequality;
3). Heat equation on the circle;
4). Weyl's equidistribution theorem;
5). Random walks;
6). Poisson summation formula;
7). Jacobi theta identities;
8). Paley–Wiener theorem;
9). Sampling theorem;
10). Heisenberg uncertainty principle;
11). Polynomial approximation;
12). Gibbs' phenomenon;
13). Central limit theorem;
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CONTENTS

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1. FOURIER SERIES

1.1. **Main definition and some examples.** We shall follow Stein-Shakarchi's book in this section. Fix $a \in \mathbb{R}, L > 0$. Let f be a continuous function on $[a, a+L]$.

Definition 1.1. *The n-th Fourier coefficient of f is defined by*

$$\hat{f}(n) = \frac{1}{L}\int_a^{a+L} f(x)e^{-2\pi inx/L}\,dx, \qquad n \in \mathbb{Z}.$$

The Fourier series of f is formally given by

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx/L}.$$

Remark: At this point, we do not say anything about the convergence of the series. The followings are examples from page 36–38 in [12]:

Example 1: Let $f(x) = x$ for $x \in [-\pi, \pi]$. We have

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\,dx.$$

Since

$$(e^{-inx})' = (-in)e^{-inx},$$

if $n \neq 0$ we have (by integration by parts)

$$\int_{-\pi}^{\pi} xe^{-inx}\,dx = \int_{-\pi}^{\pi} x\left(\frac{e^{-inx}}{-in}\right)'\,dx = x\left(\frac{e^{-inx}}{-in}\right)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(\frac{e^{-inx}}{-in}\right)\,dx = \frac{2\pi(-1)^n}{-in},$$

which gives

$$\hat{f}(n) = \frac{(-1)^{n+1}}{in}, \qquad n \neq 0.$$

If $n = 0$ then

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0.$$

Thus

$$f(x) \sim \sum_{n\neq 0} \frac{(-1)^{n+1}}{in}e^{inx}.$$

Example 2: Fix $s \in \mathbb{C}\backslash\mathbb{Z}$ and consider

$$f(x) = \frac{\pi}{\sin \pi s}e^{i(\pi-x)s}, \qquad 0 \leqslant x \leqslant 2\pi.$$

By definition, we have

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\pi}{\sin \pi s}e^{i(\pi-x)s}\overline{e^{inx}}\,dx = \frac{e^{i\pi s}}{2\sin \pi s}\int_0^{2\pi} e^{-i(n+s)x}\,dx,$$

integration by parts gives

$$\int_0^{2\pi} e^{-i(n+s)x}\, dx = \frac{e^{-i(n+s)x}}{-i(n+s)}\Big|_0^{2\pi} = \frac{e^{-2i\pi s}-1}{-i(n+s)}.$$

Apply the Euler formula $e^{ix} = \cos x + i \sin x$ (see the appendix), we get

$$\hat{f}(n) = \frac{1}{n+s}$$

and

$$f(x) \sim \sum_{n\in\mathbb{Z}} \frac{e^{inx}}{n+s}.$$

Later we shall see that the right hand side is closely related to the *Green function* of $-i\frac{d}{dx}+s$ on $\mathbb{R}/2\pi\mathbb{Z}$, where $-i\frac{d}{dx}$ is called the *one dimensional Dirac operator*.

Example 3: The trigonometric polynomial defined for $x \in [-\pi,\pi]$ by

$$D_N(x) := \sum_{n=-N}^{N} e^{inx}$$

is called the N-th *Dirichlet kernel* and is of fundamental importance in the theory (as we shall see later). A closed formula for the Dirichlet kernel is

$$D_N(x) = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}},\ \ 0<|x|<\pi; \quad D_N(x) = 2N+1,\ \ x=0. \tag{1.1}$$

This can be seen by summing the geometric progressions $\sum_{n=0}^{N}\omega^n$ and $\sum_{n=-N}^{-1}\omega^n$ with $\omega = e^{ix}$. These sums are, respectively, equal to

$$\frac{1-\omega^{N+1}}{1-\omega}, \ \text{ and } \ \frac{1-\omega^{-N-1}}{1-\omega^{-1}} - 1.$$

Their sum is then

$$\frac{\omega^{-N}-\omega^{N+1}}{1-\omega} = \frac{\omega^{-N-\frac{1}{2}}-\omega^{N+\frac{1}{2}}}{\omega^{-\frac{1}{2}}-\omega^{\frac{1}{2}}} = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}},$$

giving the desired result.

Example 4: The function $P_r(\theta)$, called the *Poisson kernel*, is defined for $\theta \in [-\pi,\pi]$ and $0 \leqslant r < 1$ by the absolutely and uniformly convergent series

$$P_r(\theta) = \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta}.$$

We know that the n-th Fourier coefficient of P_r is $r^{|n|}$ and

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2},$$

the proof is similar to the Dirichlet kernel case.

1.2. **The first question.** Let us start form the following fact: *all trigonometric polynomials are equal to their Fourier series*. In fact, if f is a degree N trigonometric polynomial, i e

$$f(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

on $[a, a+2\pi]$, then the lemma below implies

$$\hat{f}(n) = c_n, \ \ |n| \leqslant N; \quad \hat{f}(n) = 0, \ \ |n| > N.$$

Lemma 1.1. *If $m \neq n$ then*

$$\int_a^{a+2\pi} e^{imx}\overline{e^{inx}}\, dx = \int_a^{a+2\pi} e^{i(m-n)x}\, dx = 0;$$

if $m = n$ then

$$\int_a^{a+2\pi} e^{imx}\overline{e^{inx}}\, dx = \int_a^{a+2\pi} 1\, dx = 2\pi.$$

We hope that a large class of functions have Fourier series expansion. A few reflections lead us to study the following question:

Let f be a 2π-periodic function on $\mathbb{R}$. Find a natural condition on f such that all

$$\hat{f}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{e^{inx}}\, dx$$

are well defined and the following series $\{f_N(x)\}_{N\geqslant 0}$ defined by

$$f_N(x) := \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$$

converges (in a certain sense) to $f(x)$.

1.3. **Pointwise convergence.** Denote by $C^k(S^1)$ the space of all 2π-periodic C^k functions on $\mathbb{R}$, we shall use the following extension of $C^k(S^1)$.

Definition 1.2. *A 2π-periodic function f on $\mathbb{R}$ is said to be piecewise C^k $(k = 0, 1, \cdots)$ if there exists*

$$-\pi = x_0 < x_1 < \cdots < x_{m-1} < x_m = \pi, \qquad m \geqslant 1,$$

such that for each $0 < j \leqslant m-1$, $f|_{(x_j, x_{j+1})}$ extends to a C^k function on a neighborhood of $[x_j, x_{j+1}]$. We shall denote by $PC^k(S^1)$ the space of all 2π-periodic piecewise C^k functions.

Remark: Let $f \in PC^0(S^1)$. Then all its Fourier coefficients

$$\hat{f}(n) = \frac{1}{2\pi}\sum_{j=0}^{m-1}\int_{x_j}^{x_{j+1}} f(x)\overline{e^{inx}}\, dx$$

are well defined. Lemma 1.1 suggests to define:

Definition 1.3. *We call*

$$(f, g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx,$$

the inner product on $PC^0(S^1)$ and write $||f|| = (f, f)^{\frac{1}{2}}$.

Remark: It is clear that

$$\hat{f}(n) = (f, e^{inx}), \qquad \forall\ f \in PC^0(S^1),$$

which gives

$$f_N(x_0) = (f, D_N(x - x_0)). \tag{1.2}$$

We shall prove

Theorem 1.1. *If $f \in PC^1(S^1)$ then*

$$\lim_{N\to\infty} \left| f_N(x) - \frac{f(x+) + f(x-)}{2} \right| = 0. \tag{1.3}$$

Remark: For a general function $f \in PC^0(S^1)$, we have

$$\lim_{N\to\infty} \left| \frac{f_0(x) + \cdots + f_{N-1}(x)}{N} - \frac{f(x+) + f(x-)}{2} \right| = 0,$$

the proof is given in Exercise set 1 (see the end of the notes).

1.4. **Proof of Theorem 1.1 by Chernoff** [2].

1.4.1. *Chernoff identity.* Fix $x_0 \in \mathbb{R}$, assume further that f is C^1 near x_0. Then

$$g(x - x_0) = \frac{f(x) - f(x_0)}{e^{i(x-x_0)} - 1}$$

lies in $PC^0(S^1)$ and $g(y)$ is continuous near $y = 0$ (try!). Notice that

$$f(x) = (e^{i(x-x_0)} - 1)g(x - x_0) + f(x_0).$$

Change of variable $y = x - x_0$ gives the following *Chernoff identity* (try!)

$$f_N(x_0) - f(x_0) = \hat{g}(-N - 1) - \hat{g}(N).$$

1.4.2. *Bessel inequality.* Apply Lemma 1.1, we get

$$(g_N, g) = (g_N, g_N).$$

Thus we know that g_N is orthogonal to $g - g_N$, i.e.

$$(g_N, g - g_N) = 0,$$

which gives the following *Bessel inequality*

$$||g||^2 = ||g - g_N||^2 + ||g_N||^2 \geqslant ||g_N||^2 = \sum_{|n|\leqslant N} |\hat{g}(n)|^2.$$

Remark: The Bessel inequality

$$||g||^2 \geqslant \sum_{|n|\leqslant N} |\hat{g}(n)|^2$$

is true for every $g \in PC^0(S^1)$.

1.4.3. *Riemann–Lebesgue lemma.* The Bessel inequality gives the following *Riemann–Lebesgue lemma*

$$|\hat{g}(n)| \to 0, \qquad \text{as } |n| \to \infty,$$

which proves (1.3) in case f is C^1 at x.

Remark: The Riemann–Lebesgue lemma

$$|\hat{g}(n)| \to 0, \qquad \text{as } |n| \to \infty,$$

is true for every $g \in PC^0(S^1)$.

1.4.4. *Using Dirichlet kernel for the general case.* Recall that

$$f_N(x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0+x)D_N(x)\,dx.$$

Since D_N is even and $f(x_0+x)$ can be written as

$$f(x_0+x) = f_1(x) + f_2(x),$$

where

$$f_1(x) := \frac{f(x_0+x)+f(x_0-x)}{2}$$

is even and

$$f_2(x) := \frac{f(x_0+x)-f(x_0-x)}{2}$$

is odd, we have

$$f_N(x_0) = (f_1, D_N).$$

Notice that (try!) $f \in PC^1(S^1)$ implies that f_1 is C^1 near x_0, thus the previous argument gives

$$\lim_{N\to\infty} |f_N(x_0) - f_1(0)| = 0.$$

Since $f_1(0) = \frac{f(x_0+)+f(x_0-)}{2}$, the proof of Theorem 1.1 is complete.

1.5. **Mean square convergence.**

Theorem 1.2. *For every $f \in C^1(S^1)$, we have*

$$|f_N(x) - f(x)|^2 \leqslant ||f'||^2 \cdot \frac{2}{N}, \qquad \forall\, x \in \mathbb{R},$$

in particular $||f_N - f||^2 \leqslant ||f'||^2 \cdot \frac{2}{N}$

Proof. Let $N' > N$, we have

$$|f_N(x) - f_{N'}(x)| = |\sum_{|n|=N+1}^{N'} \hat{f}(n) e^{inx}| \leqslant \sum_{|n|=N+1}^{N'} |\hat{f}(n)|.$$

Notice that if $n \neq 0$ then

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\frac{e^{-inx}}{-in}\right)' dx.$$

Thus integration by parts gives

$$\hat{f}(n) = \frac{-i}{n} \cdot \widehat{f'}(n).$$

Now we have

$$|f_N(x) - f_{N'}(x)| \leqslant \sum_{|n|=N+1}^{N'} |\widehat{f'}(n)| \cdot \frac{1}{n}.$$

Thus the Cauchy-Schwarz inequality gives

$$|f_N - f_{N'}|^2 \leqslant \left(\sum_{|n|=N+1}^{N'} |\widehat{f'}(n)|^2\right) \cdot \left(\sum_{|n|=N+1}^{N'} \frac{1}{n^2}\right).$$

Since

$$\sum_{|n|=N+1}^{N'} \frac{1}{n^2} = 2\sum_{n=N+1}^{N'} \frac{1}{n^2} \leqslant 2\int_N^{\infty} \frac{dx}{x^2} = \frac{2}{N}$$

and Bessel's inequality gives

$$\sum_{|n|=N+1}^{N'} |\widehat{f'}(n)|^2 \leqslant ||f'||^2,$$

we have

$$|f_N(x) - f_{N'}(x)|^2 \leqslant ||f'||^2 \cdot \frac{2}{N},$$

thus the theorem follows from Theorem 1.1 by letting $N' \to \infty$. □

Remark: Since $\{e^{inx}\}_{n\in\mathbb{Z}}$ satisfies

$$(e^{inx}, e^{inx}) = 1, \ \ (e^{inx}, e^{imx}) = 0, \ \ n \neq m,$$

we know that $\{e^{inx}\}_{n\in\mathbb{Z}}$ is an *orthonormal family* in $C^1(S^1)$. The above theorem says that every element in $C^1(S^1)$ can be *approximated* by finite sums $\sum_{|n|\leqslant N} c_n e^{inx}$ generated by $\{e^{inx}\}_{n\in\mathbb{Z}}$, thus we know that

Theorem 1.3. *$\{e^{inx}\}_{n\in\mathbb{Z}}$ is an orthonormal basis in $C^1(S^1)$.*

Another consequence of Theorem 1.2 is the following

Theorem 1.4 (Parseval's identity)**.** $||f||^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$ *for every* $f \in C^1(S^1)$.

Remark: A similar proof (use "Week 2" exercise 5) in fact implies that *the Parseval's identity is also true on $PC^1(S^1)$*.

1.5.1. *Completion of $C^1(S^1)$*. Let us recall the following definitions

Definition 1.4. *$\{g_n\}_{n\in\mathbb{N}} \subset C^1(S^1)$ is said to be a Cauchy sequence if for every j there exists N_j such that*

$$||g_n - g_m|| < \frac{1}{j}, \qquad \forall\, n, m \geqslant N_j.$$

Two Cauchy sequences $\{g_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ are said to be equivalent if for every j there exists N_j such that

$$||g_n - h_n|| < \frac{1}{j}, \qquad \forall\, n \geqslant N_j.$$

Denote by $[\{g_n\}]$ the set of Cauchy sequences equivalent to $\{g_n\}$, we call $[\{g_n\}]$ the equivalent class of $\{g_n\}$.

One may check that (try!)

$$([\{g_n\}], [\{h_n\}]) := \lim_{n\to\infty} (g_n, h_n)$$

is well defined.

Definition 1.5. *We call the set of all equivalent classes of Cauchy sequences in $C^1(S^1)$ with the above inner product the completion of $C^1(S^1)$ and denote it by $L^2(S^1)$ or $L^2[-\pi, \pi]$.*

Remark: Notice that

$$f \mapsto [\{f, f, \cdots, \}]$$

defines an injective map from $C^1(S^1)$ to $L^2(S^1)$. Thus we may look at $C^1(S^1)$ as a subset in $L^2(S^1)$. We leave it as an *exercise* to check that $C^1(S^1)$ is dense in $L^2(S^1)$. Thus Theorem 1.2 implies

Theorem 1.5. *$L^2(S^1)$ is a* separable complex Hilbert space *with orthonormal basis $\{e^{inx}\}_{n\in\mathbb{Z}}$.*

Remark: Anther way to look at $L^2[-\pi,\pi]$ is to use the *Lebesgue integral* theory (see Appendix 2), which gives the following isomorphism (the proof can be found in [13])

$$L^2[-\pi,\pi] \simeq \{f \in \mathcal{M}[-\pi,\pi] : \int_{-\pi}^{\pi} |f|^2 dx < \infty\}/\sim,$$

where $\mathcal{M}[-\pi,\pi]$ denote the space of Lebesgue measurable complex valued functions on $[-\pi,\pi]$ and

$$f \sim g \Leftrightarrow f = g \ \ a.e. \text{ on } [-\pi,\pi].$$

1.6. **Fourier series as an eigenfunction expansion.** Let us look at the following *Dirac operator*

$$D := -i\frac{d}{dx} : f \mapsto -if',$$

on $C^\infty(S^1)$. It satisfies the following property.

Lemma 1.2. *$(Df,g) = (f,Dg)$ for every $f,g \in C^\infty(S^1)$.*

Proof. Recall that the inner product is defined by

$$(Df,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} -if'(x)\overline{g(x)}\,dx.$$

Thus the first formula follows directly from integration by parts

$$\int_{-\pi}^{\pi} f'(x)\overline{g(x)}\,dx = f(x)\overline{g(x)}\big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x)\overline{g'(x)}\,dx = -\int_{-\pi}^{\pi} f(x)\overline{g'(x)}$$

where the second identity follows since f,g are 2π-periodic. □

Remark 1: In general, a linear operator T on $C^\infty(S^1)$ is said to be *self-adjoint* if

$$(Tf,g) = (f,Tg),$$

for every $f,g \in C^\infty(S^1)$. The above lemma implies that D is self-adjoint. Moreover, we know that the square of D is the *Laplacian operator* $\Box := -\frac{d^2}{dx^2}$, that is the reason why we call D the Dirac operator.

Remark 2: We know that *all Hermitian matrices are diagonalizable with real eigenvalues*. This fact is also true for our Dirac operator D, in fact we have

$$D(e^{inx}) = n(e^{inx}),$$

we call e^{inx} the *eigenfunction of D with eigenvalue n*. Thus we may look at the Fourier series expansion of $f \in C^\infty(S^1)$ as an *eigenfunction expansion* (compare it with the eigen-theory of matrices). Since $\{e^{inx}\}$ generates $C^\infty(S^1)$, we know that D has no other eigenvalues.

Remark 3: Recall that if $s \in \mathbb{C}$ is not an eigenvalue of a matrix M then $M - s$ is invertible. Apply this fact to D, it is natural to ask whether $D+s$ is invertible in case $s \in \mathbb{C}\backslash\mathbb{Z}$. In fact, the Fourier series expansion defines the inverse of $D+s$ directly as follows

$$(D+s)^{-1}f : y \mapsto \sum \hat{f}(n)\frac{e^{iny}}{n+s}, \quad f \in C^\infty(S^1).$$

Since $\hat{f}(n) = \frac{-i}{n}\hat{f'}(n)$ and f is smooth we know that $\sum \hat{f}(n)\frac{e^{iny}}{n+s}$ also lies in $C^\infty(S^1)$. Moreover, $\hat{f}(n) = (f, e^{inx})$ gives

$$\left((D+s)^{-1}f\right)(y) = (f, G),$$

where G is defined as follows:

Definition 1.6. *For every fixed $s \in \mathbb{C}\backslash\mathbb{Z}$, $y \in \mathbb{R}$, we call*

$$G(x) := \sum_{n\in\mathbb{Z}} \frac{e^{in(x-y)}}{n+\bar{s}},$$

the y-Green function of $D+s$.

Recall that in *Example 2*, we proved that $\sum_{n\in\mathbb{Z}} \frac{e^{inx}}{n+s}$ is the Fourier series expansion of

$$f(x) := \frac{\pi}{\sin \pi s} e^{i(\pi - x)s}, \qquad 0 \leqslant x \leqslant 2\pi.$$

Thus Theorem 1.1 gives

$$\sum_{n\in\mathbb{Z}} \frac{e^{inx}}{n+s} = \frac{\pi}{\sin \pi s} e^{i(\pi-x)s}, \quad \forall\, x \in (0, 2\pi), \tag{1.4}$$

and the following crucial identity in *Eisenstein series* (see page 5 in [4])

$$\sum_{n\in\mathbb{Z}} \frac{1}{n+s} = \frac{f(0)+f(2\pi)}{2} = \pi \cot \pi s, \tag{1.5}$$

i.e. $\sum_{n\in\mathbb{Z}} \frac{e^{inx}}{n+s}$ is equal to a 2π-periodic function g such that

$$g(x) = \frac{\pi}{\sin \pi s} e^{i(\pi-x)s}, \quad \forall\, x \in (0, 2\pi)$$

and

$$g(0) = \pi \cot \pi s.$$

Now we know that G lies in $PC^\infty(S^1)$, a closed formula for G on $[0, 2\pi]$ is the following:

Theorem 1.6. *Assume that $y \in (0, 2\pi)$ and $s \in \mathbb{C}\backslash\mathbb{Z}$. We have $G(y) = \pi \cot \pi \bar{s}$,*

$$G(x) = \frac{\pi}{\sin \pi \bar{s}} e^{i(\pi - x + y)\bar{s}}, \quad \forall\, x \in (y, 2\pi]$$

and

$$G(x) = \frac{\pi}{\sin \pi \bar{s}} e^{-i(\pi + x - y)\bar{s}}, \quad \forall\, x \in [0, y).$$

Proof. The first two formulas following directly (1.4) and (1.5). The last formula follows from

$$\sum_{n\in\mathbb{Z}} \frac{e^{inx}}{n+s} = g(x+2\pi) = \frac{\pi}{\sin \pi s} e^{-i(\pi+x)s}, \quad \forall\, x \in (-2\pi, 0),$$

(think of x as $x-y$, s as $\bar{s}$). ∎

Remark 1: Since $G \in PC^\infty(S^1)$, by the mean square convergence theorem, we know that

$$||G - \sum_{|n|\leqslant N} \frac{e^{in(x-y)}}{n+\bar{s}}|| \to 0, \quad N \to \infty,$$

and

$$||G||^2 = \sum_{n\in\mathbb{Z}} \frac{1}{|n+s|^2}.$$

The above theorem gives (by a direct computation)

$$||G||^2 = \frac{\pi \sinh(2\pi \mathrm{Im}\, s)}{|\sin(\pi s)|^2 \mathrm{Im}\, s} = \frac{\pi \sinh(2\pi \mathrm{Im}\, s)}{(\cosh(2\pi \mathrm{Im}\, s) - \cos(2\pi \mathrm{Re}\, s))\mathrm{Im}\, s}.$$

Remark 2: Another way of looking at G is the following: G is the unique 2π-*periodic distribution* on $\mathbb{R}$ that solves

$$(D+\bar{s})(G) = 2\pi \sum_{k\in\mathbb{Z}} \delta_{y+2\pi k},$$

where δ_ξ is known as the Dirac distribution or *Dirac's delta function* (we will come back to it later).

Question: It is natural to ask the following questions:

1. What will happen if s goes to zero and how to define the Green function for D itself ?
2. Can you develop similar theories for D^k (try the Laplacian D^2 first)?
3. What is the relation between Green function of D and Green function of D^k.

1.7. Some applications of Fourier series.

1.7.1. *Wirtinger and Poincaré inequality.* We shall follow page 91 in [12]. The first version of the Wirtinger inequality (also called optimal one-dimensional Poincaré inequality) is

Theorem 1.7. *Let $f \in C^1(S^1)$ with $(f,1)=0$. Then*

$$||f|| \leqslant ||f'||,$$

with equality if and only if $f(x) = f_1(x) = \hat{f}(1)e^{ix} + \hat{f}(-1)e^{-ix}$.

Proof. By the proof of Theorem 1.2, we have

$$\hat{f}(n) = \frac{-i}{n} \cdot \widehat{f'}(n).$$

Apply Bessel's inequality to f', we have

$$||f'||^2 \geqslant \sum_{|n|\leqslant N} |\widehat{f'}(n)|^2 = \sum_{|n|\leqslant N} n^2 |\hat{f}(n)|^2.$$

Since $(f,1)=0$ implies that $\hat{f}(0)=0$, thus the Parseval's identity gives

$$||f||^2 = \lim_{N\to\infty} \sum_{0<|n|\neq N} |\hat{f}(n)|^2 \leqslant ||f'||^2,$$

with inequality if and only if $\hat{f}(n) = 0$ for all $|n| > 1$, i.e. $f = f_1$. □

Remark: Notice that $|f'| = |Df|$, thus the above theorem gives

$$||Df|| \geqslant ||f||,$$

in case $f \in C^\infty(S^1)$ with $(f, 1) = 0$. The above identity is formally equivalent to that all non-zero eigenvalues, say λ, of D satisfy

$$|\lambda| \geqslant 1.$$

Theorem 1.7 also implies

Proposition 1.1. *Let $f \in C^1(S^1)$ with $(f, 1) = 0$. Then for every $g \in C^1(S^1)$, we have*

$$|(f, g)| \leqslant ||f|| \cdot ||g'||.$$

Proof. Notice that

$$(f, g) = (f, g - \hat{g}(0)).$$

Thus the theorem follows from Cauchy–Schwarz inequality and Theorem 1.7. □

The second version of the Wirtinger inequality

Theorem 1.8. *Let f be a C^1 function in a neighborhood of $[0, \pi]$ such that $f(0) = f(\pi) = 0$. Then*

$$\int_0^\pi |f(x)|^2 \, dx \leqslant \int_0^\pi |f'(x)|^2 \, dx,$$

with equality if and only if $f(x) = A \sin x$.

Proof. Check (try!) that $f|_{[0,\pi]}$ extends to an *odd* function (still denote it by f) in $C^1(S^1)$, in particular, $(f, 1) = 0$, thus Theorem 1.7 applies. □

Remark: The above proof of the Wirtinger inequality also applies to the famous *isoperimetric inequality* (see page 103 in [12]). A natural high diemsional generalization of the *convex version of the isoperimetric inequality* is the classical Brunn–Minkowski inequality (see [7]).

Theorem 1.9 (Brunn–Minkowski Theorem)**.** *Let A_0, A_1 be two compact convex sets in $\mathbb{R}^n$ with non-empty interior. Then*

$$|A_0 + A_1|^{\frac{1}{n}} \geqslant |A_0|^{\frac{1}{n}} + |A_1|^{\frac{1}{n}},$$

where $|A|$ denotes the Lebesgue measure (volume) of A and

$$A_0 + A_1 := \{x + y \in \mathbb{R}^n : x \in A_0, \ y \in A_1\}$$

is called the Minkowski sum of A_0 and A_1.

Remark: The above Brunn–Minkowski inequality is also true for every non-empty compact sets A_0 and A_1 (this general version is proved by Lazar Lyusternik in 1935), which can be seen as a generalization of the usual isoperimetric inequality.

1.7.2. *Heat equation on the circle.* We shall follow page 61–64 in [3].

The derivation of the heat equation is based on Newton's law of cooling, which states that the *flux* of heat across a point x_0 is proportional to the gradient of the temperature at x_0 (this is an experimental fact that is well verified for moderate temperature gradients, the full experimental relationship between flux and gradient is very complicated). This means that the amount of heat that flows past x_0 from left to right in a short time $[t_0, t_0 + \delta t]$ is approximately

$$-c_1 \cdot u_x(x_0, t_0) \cdot \delta t$$

with a positive constant c_1 depending upon the meterial (the minus sign is present because *heat flows from the hotter place to the cooler*). To proceed, the net amount of heat flowing *out* of a small interval $[x_0 - \delta x, x_0 + \delta x]$ in a short time $[t_0, t_0 + \delta t]$ is therefore

$$(-c_1 \cdot u_x(x_0 + \delta x, t_0) \cdot \delta t) - (-c_1 \cdot u_x(x_0 - \delta x, t_0) \cdot \delta t),$$

or approximately so. This can be computed in a second way: It is, in fact, proportional to the product of the length of the interval and the (average) decrease of the temperature inside. The constant of proportionality is the "specific heat" of the conducting material. Therefore,

$$-c_2 u_t(x_0, t_0) 2\delta x \cdot \delta t = (-c_1 \cdot u_x(x_0 + \delta x, t_0) \cdot \delta t) - (-c_1 \cdot u_x(x_0 - \delta x, t_0) \cdot \delta t),$$

or approximately so. Letting δx and δt go to zero, you find

$$u_t(x_0, t_0) = \frac{c_1}{c_2} u_{xx}(x_0, t_0).$$

We can replace the time coordinate t by $t = cT$, where c is another positive constant. If we set $U(x, T) = u(x, cT)$, then

$$U_T(x, T) = c u_t(x, cT) = \frac{cc_1}{c_2} u_{xx}(x, cT) = \frac{cc_1}{c_2} U_{xx}(x, T).$$

Choose c such that $2cc_1 = c_2$, it is enough to study the following standard heat equation

$$u_t = \frac{1}{2} u_{xx}.$$

In this section, we shall study the above heat equation on the circle $\mathbb{R}/\mathbb{Z}$ (of length one, i.e. the temperature is one-periodic with respect to x). Because u is supposed to be the temperature, it is natural to conjecture that the whole solution is determined by the temperature

$$f(x) := u(x, 0)$$

at $t = 0$. Since u is one-periodic with respect to x, we can formally write

$$u(x, t) = \sum_{n \in \mathbb{Z}} c_n(t) e^{2in\pi x},$$

with

$$c_n(t) = \int_0^1 u(x, t) e^{-2in\pi x} \, dx.$$

It is enough to compute $c_n(t)$.

Lemma 1.3. *Each $c_n(t)$ satisfies*

$$c_n(0) = \int_0^1 f(x)e^{-2in\pi x}\,dx$$

and

$$c_n'(t) = -2\pi^2 n^2 c_n(t).$$

Proof. The first identity is just the initial condition. For the second identity, notice that

$$c_n'(t) = \int_0^1 u_t(x,t)e^{-2in\pi x}\,dx = \frac{1}{2}\int_0^1 u_{xx}(x,t)e^{-2in\pi x}\,dx.$$

Integration by parts gives

$$\int_0^1 u_{xx}(x,t)e^{-2in\pi x}\,dx = \int_0^1 u(x,t)(e^{-2in\pi x})_{xx}\,dx = -4\pi^2 n^2 \int_0^1 u(x,t)e^{-2in\pi x}\,dx.$$

Thus the lemma follows. □

Solving the above ODE gives

$$c_n(t) = \left(\int_0^1 f(y)e^{-2in\pi y}\,dy\right) e^{-2\pi^2 n^2 t},$$

thus

$$u(x,t) = \sum \left(\int_0^1 f(y)e^{-2in\pi y}\,dy\right) e^{-2\pi^2 n^2 t} e^{2in\pi x} = \int_0^1 \theta(x-y, 2\pi i t)\, f(y)\,dy$$

where θ denotes the classical Riemann theta function defined by

$$\theta(z,t) := \sum_{n\in\mathbb{Z}} e^{\pi i(n^2 t + 2nz)}, \quad z \in \mathbb{C},\ t \in \mathbb{H} := \{t \in \mathbb{C} : \operatorname{Im} t > 0\}. \tag{1.6}$$

One may check that (try!) θ is holomorphic on $\mathbb{C} \times \mathbb{H}$.

Definition 1.7. *We call θ the Jacobi theta function and*

$$h(x,y,t) := \theta(x-y, 2\pi i t), \quad x, y \in \mathbb{R},\ t > 0,$$

the heat kernel on the circle.

The heat kernel above solves the heat equation in the following sense

Theorem 1.10. *For $f \in C^2(\mathbb{R}/\mathbb{Z})$, put*

$$u(x,t) = \int_0^1 h(x,y,t)\, f(y)\,dy.$$

Then we have

1) *u is one-periodic with respect x and is smooth on $\mathbb{R} \times (0,\infty)$;*
2) *$u_t = \frac{1}{2}u_{xx}$ on $\mathbb{R} \times (0,\infty)$;*
3) $\lim_{t\to 0+} \sup_{x\in\mathbb{R}} |u(x,t) - f(x)| = 0$.

Proof. It is easy to check 1) and 2) since $e^{-2\pi^2 n^2 t}$ is a rapidly decreasing function of n. 3) follows from

$$|u(x,t) - f(x)| \leqslant \sum_{n\in\mathbb{Z}} (1 - e^{-2\pi^2 n^2 t}) |\hat{f}(n)|,$$

and

$$|\hat{f}(n)| = |(2\pi i n)^{-2} \widehat{f''}(n)| \leqslant (2\pi n)^{-2} \int_0^1 |f''(x)|\, dx.$$

□

Remark: The above theorem is also true for $f \in C^0(\mathbb{R}/\mathbb{Z})$, for the proof and the uniqueness of u, see page 64–65 in [3].

1.7.3. *Weyl's equidistribution theorem.* We shall follow page 106–112 in [12], for related results, see page 54–56 in [3]. A basic postulate of statistical mechanics is the so called *ergodic principle* of Boltzman and Gibbs, which states that the time average of a mechanical quantity should be the same as its phase average; see Ford and Uhlenbeck [6] (page 9–13) for a nice discussion of such matters. A simple instance of this phenomenon can be seen in the following model due to Weyl in 1916.

As *phase space*, bring in the circle $\mathbb{R}/\mathbb{Z}$, pick a number $0 < \gamma < 1$, and look at the *rotation*

$$x \mapsto x_1 := x + \gamma$$

with addition modulo 1. The *trajectory* of the phase point $x_0 = x$ is the arithmetic sequence

$$x_0 = x,\ x_1 = x + \gamma,\ \cdots,\ x_n = x + n\gamma,\ \cdots,$$

considered modulo 1. A *mechanical quantity* is a function $f \in PC^0(\mathbb{R}/\mathbb{Z})$. Its *time average* is

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} f(x_k)}{n},$$

assuming this limit to exist, while its *phase everage* is just the arithmetic mean

$$\int_0^1 f(x)\, dx.$$

Weyl proved that

Theorem 1.11. *If γ is irrational then*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} f(k\gamma)}{n} = \int_0^1 f(x)\, dx,$$

for every $f \in C^0(\mathbb{R}/\mathbb{Z})$.

Proof. Use Fejér's theorem to approximate f uniformaly by trigonometric polynomials (see Exercise set 1), it is enough to prove the theorem for $f = e^{2\pi i m x}$. If $m = 0$ then both sides are 1. If $m \neq 0$ then

$$\sum_{k=0}^{n-1} f(k\gamma) = \sum_{k=0}^{n-1} a^k, \quad a := e^{2\pi i m\gamma}.$$

Since γ is irrational, we know that $a \neq 1$ and

$$\sum_{k=0}^{n-1} a^k = \frac{1-a^n}{1-a}$$

is bounded by $2|1-a|^{-1}$. Thus

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} f(k\gamma)}{n} = 0 = \int_0^1 f(x)\,dx.$$

The proof is complete. □

Remark: Fix two real numbers $a < b$ with $b - a < 1$, consider the following one-periodic indicator function defined by

$$1_{[a,b]+\mathbb{Z}}(x) := 1 \text{ if } x + n \in [a,b],$$

for some ineteger n and $1_{[a,b]+\mathbb{Z}}(x) := 0$ otherwise. One may prove that the above theorem also applies to $1_{[a,b]+\mathbb{Z}}$: the idea is to approximate $1_{[a,b]+\mathbb{Z}}$ above and below by continuous functions f^+ and f^- so as to make $\int_0^1 (f^+ - f^-)\,dx$ small and use the above theorem to f^+ and f^- respectively. Thus we get: if γ is irrational then

$$\lim_{n\to\infty} \frac{\#\{k < n : k\gamma \in [a,b] + \mathbb{Z}\}}{n} = b - a,$$

and we say that $\{k\gamma + \mathbb{Z}\}$ is *equidistributed* in $\mathbb{R}/\mathbb{Z}$. A generalization of this fact is the following Weyl's criterion

Theorem 1.12. *Let $\{\xi_n\}_{n=0}^{\infty}$ be a sequence of real number. Then $\{\xi_n + \mathbb{Z}\}$ is equidistributed in $\mathbb{R}/\mathbb{Z}$ if and only if for every nonzero integer k,*

$$\lim_{N\to\infty} \frac{\sum_{n=0}^{N} e^{2\pi i k \xi_n}}{N} = 0.$$

Proof. $\Rightarrow$: Since equidistributive property of $\{\xi_n + \mathbb{Z}\}$ is equivalent to that for every one-periodic indicator function f,

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} f(\xi_n)}{n} = \int_0^1 f(x)\,dx.$$

Approximating $f \in C^0(S^1)$ by one-periodic indicator function, we know that the above identity is also true for every $f \in C^0(S^1)$. Now it is enough to apply it to $f(x) = e^{2\pi i k x}$.

$\Leftarrow$: Follows by a similar argument as in the proof of equidistributive property of $\{k\gamma + \mathbb{Z}\}$. □

1.8. **Several dimensional Fourier series.** We shall follow page 81–85 in [3].

Definition 1.8. *A function f on $\mathbb{R}^d$ is said to be $\mathbb{Z}^d$-invariant if*

$$f(x+k)=f(x), \qquad \forall\, k\in\mathbb{Z}^d.$$

We call a $\mathbb{Z}^d$-invariant function a function on the standard torus T^d.

We shall denote by $L^2(T^d)$ the completion of the space $C^\infty(T^d)$ of smooth $\mathbb{Z}^d$-invariant functions with respect to the following inner product

$$(f,g):=\int_0^1\cdots\left(\int_0^1 f(x_1,\cdots,x_d)\overline{g(x_1,\cdots,x_d)}\,dx_1\right)dx_2\cdots dx_d,\quad f,g\in C^\infty(T^d).$$

Since finite $\mathbb{C}$-linear combinations of functions in $\{1|_Q\}$, where Q denotes an arbitray n-cube, are dense in $L^2(T^d)$ and

$$1|_Q(x)=1|_{Q_1}(x_1)\cdots 1|_{Q_d}(x_d), \qquad Q:=Q_1\times\cdots\times Q_d,$$

we know that

$$e_Z(x):=e^{2\pi i Z\cdot x},\quad Z\in\mathbb{Z}^d$$

defines an orthonormal basis of $L^2(T^d)$, which gives

1.8.1. *Fourier series on a standard Torus.*

Theorem 1.13. *Every $f\in L^2(T^d)$ satisfies*

$$\lim_{N\to\infty}||f-\sum_{Z\in[-N,N]^d}\hat f(Z)\,e_Z||\to 0,$$

(later we shall write the above identity as $f=\sum_{Z\in\mathbb{Z}^d}\hat f(Z)e_Z$) where

$$\hat f(Z):=(f,e_Z),$$

denotes the Z-th Fourier coefficient of f.

1.8.2. *Application to Random walks.* Pólya discovered a very beautiful application of several dimensional Fourier series to "random walks". Think of a particle moving on the d-dimensional lattice $\mathbb{Z}^d$ according to the following rule. The particle starts at time 0 at the origin and moves at time $n\geqslant 1$ by a unit step u_n to a neighborhood lattice point; for example, if $d=3$, the possible steps are

$$u=(\pm 1,0,0),\ \ (0,\pm,0),\ \ (0,0,\pm 1), \qquad |u|=1.$$

The position of the particle at time $n\geqslant 1$ is the sum of the individual steps: $s_n=u_1+\cdots+u_n$. The step u_n is statistically independent of the preceding steps $u_j : j<n$ and the possible steps are equally likey at each stage. This means that

$$P(u_1=\hat u_1,\cdots,u_n=\hat u_n)=P(u_1=\hat u_1)\times\cdots\times P(u_n=\hat u_n)=(2d)^{-n}$$

for any fixed unit steps $\hat u_1,\cdots,\hat u_n$, in which $P(E)$ means "the probability of the event E".

The problem is to compute $P(s_n=Z)$ and to study the behavior of s_n for $n\to\infty$.

Pólya's idea is to think of $P(s_n = Z)$ as the Fourier coefficient $\hat{f}(Z)$ of a function $f \in L^2(T^n)$:

$$f(x) = \sum P(s_n = Z) e_Z(x) = \sum_{Z \in \mathbb{Z}^d} P(s_n = Z) e^{2\pi i Z \cdot x}.$$

The sum is just the "expectation" or "mean value" of $e^{2\pi i s_n \cdot x}$ and is easily computed using the independence of the individual steps. In fact, notice that

$$P(s_n = Z) = \sum_{\hat{u}_1 + \cdots + \hat{u}_n = Z} P(u_1 = \hat{u}_1) \times \cdots \times P(u_n = \hat{u}_n) = \sum_{\hat{u}_1 + \cdots + \hat{u}_n = Z} (2d)^{-n}.$$

Thus

$$P(s_n = Z) e^{2\pi i Z \cdot x} = \sum_{\hat{u}_1 + \cdots + \hat{u}_n = Z} (2d)^{-n} e^{2\pi i \, (\hat{u}_1 + \cdots + \hat{u}_n) \cdot x},$$

which gives

$$f(x) = \sum_{Z \in \mathbb{Z}^d} \sum_{\hat{u}_1 + \cdots + \hat{u}_n = Z} (2d)^{-n} e^{2\pi i \, (\hat{u}_1 + \cdots + \hat{u}_n) \cdot x} = \left((2d)^{-1} \sum_{|u|=1} e^{2\pi i u \cdot x} \right)^n.$$

Since

$$\sum_{|u|=1} e^{2\pi i u \cdot x} = 2 \left(\cos 2\pi x_1 + \cdots + \cos 2\pi x_d \right),$$

put

$$f_d(x) := \frac{\cos 2\pi x_1 + \cdots + \cos 2\pi x_d}{d},$$

we get

$$f(x) = f_d(x)^n.$$

Thus Theorem 1.13 gives the following *Pólya's formula*

$$P(s_n = Z) = \hat{f}(Z) = (f_d^n, e_Z).$$

In particular,

$$P(s_n = 0) = \int_{[0,1]^d} f_d(x)^n \, dx_1 \cdots dx_d.$$

Since $|f_d| \leqslant 1$, the expected number of times the particle visits the origin can be expressed as

$$\sum_{n=0}^{\infty} P(s_n = 0) = \lim_{\varepsilon \to 1} \sum_{n=0}^{\infty} \varepsilon^n P(s_n = 0) = \lim_{\varepsilon \to 1} \int_{[0,1]^d} \sum_{n=0}^{\infty} \varepsilon^n f_d(x)^n \, dx_1 \cdots dx_d.$$

Since

$$\lim_{\varepsilon \to 1} \int_{[0,1]^d} \sum_{n=0}^{\infty} \varepsilon^n f_d(x)^n = \lim_{\varepsilon \to 1} \int_{[0,1]^d} (1 - \varepsilon f_d)^{-1} = \int_{[0,1]^d} (1 - f_d)^{-1},$$

we get

$$\sum_{n=0}^{\infty} P(s_n = 0) = \int_{[0,1]^d} \frac{1}{1 - \frac{\cos 2\pi x_1 + \cdots + \cos 2\pi x_d}{d}} \, dx_1 \cdots dx_d.$$

By the definition of e^x in Appendix 1 and the Euler formula, we get

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots ,$$

which implies that

$$\frac{x^2}{3} \leqslant 1 - \cos x \leqslant \frac{x^2}{2},$$

when $|x|$ is small enough. Thus (leave as an exercise) $\sum_{n=0}^{\infty} P(s_n = 0) < \infty$ if and only if

$$\int_{[0,\varepsilon]^d} \frac{1}{x_1^2 + \cdots + x_d^2} \, dx_1 \cdots dx_d < \infty,$$

which is equivalent to $d \geqslant 3$ (by using the polar coordinate). Pólya used this to prove a very striking fact about the ultimate begavior of the walk.

Theorem 1.14. *If $d \geqslant 3$ then*

$$P(\lim_{n\to\infty} |s_n| = \infty) = 1.$$

If $d \leqslant 2$ then

$$P(s_n = 0 \text{ infinitely often}) = 1.$$

Proof. If $d \geqslant 3$ then we know that the expected number, say P, of times the particle visits the origin is less than infinity. Denote by p_n the probability of n actural number of visits. We know that

$$P = p_\infty \times \infty + \sum_{n=0}^{\infty} n p_n < \infty, \quad p_\infty + \sum_{n=0}^{\infty} p_n = 1.$$

Thus we must have $p_\infty = 0$ and the actual number of visits is less than infinity with probability 1, and since the origin is not special in any way, the same must be true for every lattice point in $\mathbb{Z}^d$. This means that for any $R < \infty$, the particle ultimately stops visiting the ball $|Z| < R$, and that is the same as to say

$$P(\lim_{n\to\infty} |s_n| = \infty) = 1.$$

Now let us assume that $d \leqslant 2$. At time $n = 1$, the particle steps to one of the $2d$ nearest neighbors of the origin. The problem is to check that the probability p of ultimately returning to the origin is 1. In fact, the probability of visiting the origin m or more times (including the visit at time $n = 0$) is p^{m-1}. Thus the probability of precisely m visits is

$$p^{m-1} - p^m = p^{m-1}(1-p).$$

If $p < 1$ then $p^\infty = 0$ and the expected number of visits would be

$$\infty \times 0 + \sum_{m=1}^{\infty} m p^{m-1}(1-p) = (1-p)^{-1} < \infty,$$

contradicting the evaluation that $\sum_{n=0}^{\infty} P(s_n = 0) = \infty$. The proof is finished; for additional information on the subject, see Feller [5], pp 342–371. □

1.8.3. *Fourier series on a two dimensional Torus.* Pick numbers $a \in \mathbb{R}$, $b > 0$ and introduce the "non-standard" lattice $L \subset \mathbb{R}^2$ of all points of the form

$$\omega = j(1,0) + k(a,b), \qquad (j,k) \in \mathbb{Z}^2.$$

Definition 1.9. *A function f on $\mathbb{R}^2$ is said to be L-periodic if*

$$f(x + (1,0)) = f(x) = f(x + (a,b)),$$

for every $x \in \mathbb{R}^2$.

Definition 1.10. *We call the set of all points $\omega' \in \mathbb{R}^2$ such that*

$$\omega' \cdot \omega \in \mathbb{Z},$$

for every $\omega \in L$ the dual lattice of L, and denote it by L'.

Exercise: Check that L' is the lattice of points

$$\omega' = j(1, -\frac{a}{b}) + k(0, \frac{1}{b}), \qquad (j,k) \in \mathbb{Z}^2.$$

$L' = L$ if and only if $a \in \mathbb{Z}$ and $b = 1$.

Exercise: Check that

$$e_\gamma(x) := e^{2\pi i \gamma \cdot x}$$

is L-periodic if and only if $\gamma \in L'$.

Remark: One may look at the torus

$$T_L := \mathbb{R}^2 / L$$

by identifying opposite sides of the following "fundamental cell"

$$F_L := \{t(1,0) + s(a,b) : 0 \leqslant t, s \leqslant 1\}.$$

Denote by $C^\infty(T_L)$ the space of smooth L-periodic functions on $\mathbb{R}^2$. Let $L^2(T_L)$ be the completion of $C^\infty(T_L)$ with respect to the following inner product

$$(f,g)_L := \int_{F_L} f(x)\overline{g(x)}\, dx_1 \cdots dx_n, \qquad f, g \in C^\infty(T_L).$$

Then we have the following generalization of the standard torus Fourier series expansion.

Theorem 1.15. *Every $f \in L^2(T_L)$ has the following orthogonal decomposition*

$$f = \sum_{\gamma \in L'} \hat{f}(\gamma) e_\gamma,$$

where

$$\hat{f}(\gamma) := (f, e_\gamma)_L.$$

Moreover, the following Plancherel identity holds

$$||f||_L^2 = b \cdot \sum_{\gamma \in L'} |\hat{f}(\gamma)|^2.$$

Proof. Think of f and e_γ as functions of $y_1 = x_1 - \frac{a}{b}x_2$ and $y_2 = \frac{1}{b}x_2$. This will bring you back to the standard torus case. □

Exercise: Try to write down the details of the proof of the above theorem (notice that b is the area of F_L; if f is Z-periodic then

$$g(y) := f(y_1 + ay_2, by_2),$$

is $\mathbb{Z}^n$-invariant).

Exercise: Try to develop similar Fourier series theory for general high dimensional torus.

2. Fourier transform

Recall that if a smooth function on $\mathbb{R}$ is $2L$-periodic then the following Fourier series expansion holds

$$f(x) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2L} \int_{-L}^{L} f(y) e^{-2\pi i n \frac{y}{2L}}\, dy \right) e^{2\pi i n \frac{x}{2L}}.$$

A small change in viewpoint leads at once to the Fourier integral: the idea is that the right hand side is really a Riemann sum over a subdivision with spacing $\frac{1}{2L}$, and with any luck, it should approximate the integral

$$f(x) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y) e^{-2\pi i y\gamma}\, dy \right) e^{2\pi i x\gamma}\, d\gamma$$

as L goes to ∞. This does not make too much sense for a periodic function f (the integral cannot converge well), but it does suggest that something can be done to recover a nice function f from its *Fourier integral (or transform)*:

$$\hat{f}(\gamma) := \int_{-\infty}^{\infty} f(y) e^{-2\pi i y\gamma}\, dy$$

via the *inverse Fourier integral*

$$\check{\hat{f}} = \int_{-\infty}^{\infty} \hat{f}(\gamma) e^{2\pi i x\gamma}\, d\gamma.$$

The purpose of the next two sections is to put this formal discussion on a solid mathematical foundation.

2.1. **Fourier transform on the Schwartz space.**

Definition 2.1. *By the Schwartz space, say $\mathcal{S}$, on $\mathbb{R}$, we mean the space of all smooth functions, say f, on $\mathbb{R}$ such that*

$$\sup_{x\in\mathbb{R}} |x|^k |f^{(l)}(x)| < \infty,$$

for every non-negative integers k, l, where $f^{(l)}$ denotes the l-th order derivative of f.

Examples: $e^{-x^2} \in \mathcal{S}$ but $(1 + x^2)^{-1}$ does not belong to $\mathcal{S}$; nor does $e^{-|x|}$, but for a different reason (try to verify this statement).

Definition 2.2. *The Fourier transform of a function $f \in \mathcal{S}$ is defined by*

$$\hat{f}(\gamma) := \int_{-\infty}^{\infty} f(y) e^{-2\pi i y\gamma}\, dy.$$

The inverse Fourier transform of a function $g \in \mathcal{S}$ is defined by

$$\check{g}(x) = \int_{-\infty}^{\infty} g(\gamma) e^{2\pi i x\gamma}\, d\gamma.$$

Remark: We have $\hat{f}(x) = \check{f}(-x)$.

The Fourier transform interchanges convolutions with pointwise products. Moreover, we have

Proposition 2.1. *If $f \in \mathcal{S}$ then*

1) $\widehat{f(x+h)} = \hat{f}(\gamma) e^{2\pi i h\gamma}$ *whenever* $h \in \mathbb{R}$;
2) $\widehat{f(x)e^{-2\pi i h x}} = \hat{f}(\gamma + h)$ *whenever* $h \in \mathbb{R}$;
3) $\widehat{f(\delta x)} = \delta^{-1} \hat{f}(\delta^{-1}\gamma)$ *whenever* $\delta > 0$;
4) $\widehat{f'(x)} = 2\pi i \gamma \hat{f}(\gamma)$;
5) $\widehat{-2\pi i x f(x)} = \frac{d}{d\gamma} \hat{f}(\gamma)$.

Proof. We only prove 4) and leave the others as exercises (see page 136–137 in [12]). Integration by parts gives

$$\int_{-N}^{N} f'(x) e^{-2\pi i x\gamma}\, dx = f(x) e^{-2\pi i x\gamma}\Big|_{-N}^{N} + 2\pi i\gamma \int_{-N}^{N} f(x) e^{-2\pi i x\gamma}\, dx,$$

so letting N goes to infinity gives 4). □

Example: The above theorem can be used to prove the following *crucial identity* in Fourier transform:

Proposition 2.2. *The Gaussian function $e^{-\pi x^2}$ is the fixed point of the Fourier transform, more precisely, we have*

$$\widehat{e^{-\pi x^2}} = e^{-\pi\gamma^2}.$$

Proof. Put $f(x) = \widehat{e^{-\pi x^2}}$. Then we have

$$f'(x) = -2\pi x f(x)$$

Thus 4) and 5) in the above Proposition give

$$\frac{d}{d\gamma} \hat{f}(\gamma) = \widehat{i f'(x)} = -2\pi\gamma \hat{f}(\gamma).$$

Thus formally we have $\frac{d}{d\gamma}\ln \hat{f}(\gamma) = 2\pi\gamma$ and $\hat{f}(\gamma) = ce^{-\pi\gamma^2}$ (for a rigorous proof, one may compute the derivative of $\hat{f}(\gamma)e^{-\pi\gamma^2}$ and show that it vanishes). Now it is enough to prove that $c = 1$. Notice that

$$c = \hat{f}(0) = \int_{\mathbb{R}} e^{-\pi x^2}\, dx.$$

Thus $c = 1$ follows from the Gaussian integral formula, see Week 4–Exercise 1. □

Remark 1: In general, it is also natural to study eigenvalue and eigenvectors of the Fourier transform (see Exercise set 4, see also a recent result of Rodgers and Tao [16] on non-negativity of the De Bruijn–Newman constant).

Remark 2: The above proposition together with 3) imply that for every $s > 0$,

$$\widehat{e^{-\pi x^2 s}} = s^{-\frac{1}{2}} e^{-\pi\gamma^2/s}, \tag{2.1}$$

we shall use this identity to prove the *Jacobi Theta Identity*.

The fact that Fourier transform interchanges differentiation and multiplication can be used to prove (try!) the following result.

Theorem 2.1. *If $f \in \mathcal{S}$ then $\hat{f} \in \mathcal{S}$.*

Definition 2.3. *For every $f, g \in \mathcal{S}$, we call*

$$f \star g : x \mapsto \int_{-\infty}^{\infty} f(x-y)g(y)\, dy,$$

the convolution of f and g.

Proposition 2.3. *If $f, g, h \in \mathcal{S}$ then*

1) $f \star g \in \mathcal{S}$;
2) $f \star g = g \star f$;
3) $\widehat{f \star g} = \hat{f}\hat{g}$;
4) $(f \star g) \star h = f \star (g \star h)$;
5) $(f \star g)' = f' \star g = g' \star f$.

Proof. Exercise (see page 142–143 in [12]). □

2.2. **Classical Poisson summation formula.** We need a lemma to state the classical Poisson summation formula.

Lemma 2.1. *Let f be a continuous function on $\mathbb{R}$ such that*

$$|f(x)| \leqslant C(1+x^2)^{-1}.$$

Then

$$f_T(x) := \sum_{k\in\mathbb{Z}} f(x + kT),$$

defines a T-periodic continuous function on $\mathbb{R}$ such that

$$|f_T(x) - f(x)| \leqslant C\frac{\pi^2}{T^2}, \quad \forall \ |x| \leqslant \frac{T}{2}.$$

Proof. Put

$$f_N(x) := \sum_{|k|\leqslant N} f(x+kT).$$

Then we know that f_N converges uniformly to f_T. Thus f_T is continuous and obviously f_T is T-periodic. The final estimate follows from

$$|f_T(x) - f(x)| \leqslant \frac{2C}{T^2}\sum_{k=1}^{\infty}\frac{1}{(k-\frac{1}{2})^2} = C\frac{\pi^2}{T^2} \ \forall \ |x| \leqslant \frac{T}{2}.$$

□

Remark: If $f \in \mathcal{S}$ then every n-th derivative $f^{(n)}$ of f fits the above lemma, thus f_T is smooth and T-periodic. Apply the Fourier series expansion, we get

$$f_T(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n\frac{x}{T}},$$

where

$$c_n = \frac{1}{T}\int_{|y|<\frac{T}{2}} f_T(y)e^{-2\pi i n\frac{y}{T}}\, dy.$$

Notice that

$$\int_{|y|<\frac{T}{2}} f_T(y)e^{-2\pi i n\frac{y}{T}}\, dy = \sum_{k\in\mathbb{Z}}\int_{|y|<\frac{T}{2}} f(y+kT)e^{-2\pi i n\frac{y}{T}}\, dy = \int_{\mathbb{R}} f(y)e^{-2\pi i n\frac{y}{T}}\, dy = \hat{f}\left(\frac{n}{T}\right),$$

which gives

Theorem 2.2. *If $f \in \mathcal{S}$ then*

$$f_T(x) = \frac{1}{T}\sum_{n\in\mathbb{Z}}\hat{f}\left(\frac{n}{T}\right)e^{2\pi i n\frac{x}{T}}.$$

In particular, we have the following Poisson summation formula (just take $T=1,\ x=0$)

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}}\hat{f}(n) \tag{2.2}$$

Application to Jacobi theta identities: Apply the above theorem to

$$f(x) = e^{\pi i(x^2t+2xz)}, \quad z,t\in\mathbb{C},\ \operatorname{Im} t > 0,$$

by (2.1), we get

$$\hat{f}(\lambda) = (-it)^{-\frac{1}{2}}e^{-i\pi\frac{(\lambda-z)^2}{t}}.$$

Thus by the Poisson summation formula and the definition of theta function in (1.6), we get

$$\theta(z,t) = (-it)^{-\frac{1}{2}}e^{-\frac{i\pi z^2}{t}}\theta\left(\frac{z}{t}, -\frac{1}{t}\right). \tag{2.3}$$

The readers can easily check the remaining theta identities

$$\theta(z,t+1)=\theta\left(z+\frac{1}{2},t\right), \quad \theta(z+1,t)=\theta(z,t), \quad \theta(z,t)=e^{\pi i(t+2z)}\theta(z+t,t). \tag{2.4}$$

Remark: Notice that

$$f_T(x)=\sum_{k\in\mathbb{Z}} e^{\pi i\left((x+kT)^2t+2(x+kT)z\right)} := \theta(z,t;x,T),$$

then the above theorem implies

$$T\cdot\theta(z,t;x,T)=\sum_{n\in\mathbb{Z}}(-it)^{-\frac{1}{2}}e^{-i\pi\frac{\left(\frac{n}{T}-z\right)^2}{t}}e^{2\pi in\frac{x}{T}}=(-it)^{-\frac{1}{2}}e^{-\frac{i\pi z^2}{t}}\theta\left(\frac{z}{t}+x,-\frac{1}{t};0,\frac{1}{T}\right). \tag{2.5}$$

2.3. **Fourier inversion formula and Plancherel identity.** The Poisson summation formula implies the following result (for a direct *Fourier series expansion* proof, see page 89 in [3]).

Theorem 2.3. *Every f in $\mathcal{S}$ satisfies the Fourier inversion formula*

$$f(x)=\int_{-\infty}^{\infty}\hat{f}(\gamma)e^{2\pi ix\gamma}\,d\gamma$$

and the Plancherel identity

$$\int_{-\infty}^{\infty}|f(x)|^2\,dx=\int_{-\infty}^{\infty}|\hat{f}(\gamma)|^2\,d\gamma.$$

Proof. By the definiton of Riemann integral, we have

$$\int_{-\infty}^{\infty}\hat{f}(\gamma)e^{2\pi ix\gamma}\,d\gamma=\lim_{T\to\infty}\frac{1}{T}\sum_{n\in\mathbb{Z}}\hat{f}\left(\frac{n}{T}\right)e^{2\pi in\frac{x}{T}}.$$

By Theorem 2.2, the above sum is equal to

$$f_T(x)=\sum_{k\in\mathbb{Z}}f(x+kT).$$

Thus we have

$$\int_{-\infty}^{\infty}\hat{f}(\gamma)e^{2\pi ix\gamma}\,d\gamma-f(x)=\lim_{T\to\infty}\sum_{k\neq 0}f(x+kT),$$

but obviously the above limit is zero since $f\in\mathcal{S}$. Now let us prove the Plancherel identity. By Theorem 2.2 and the Plancherel identity for the Fourier series (recall that $\{e^{2\pi inx/T}\}$ is an orthogonal basis of $L^2[-T/2,T/2]$), we have

$$\int_{|x|<\frac{T}{2}}|f_T(x)|^2\,dx=\frac{1}{T}\sum_{n\in\mathbb{Z}}\left|\hat{f}\left(\frac{n}{T}\right)\right|^2\to\int_{-\infty}^{\infty}|\hat{f}(\gamma)|^2\,d\gamma, \text{ as } T\to\infty.$$

The left hand side goes to $\int_{-\infty}^{\infty}|f(x)|^2\,dx$ as T goes to infinity, thus the theorem follows. □

Denote by

$$(f,g) := \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx,$$

the inner product space structure on $\mathcal{S}$. Put

$$||f|| := (f,f)^{\frac{1}{2}}.$$

Definition 2.4. *We call the set of all equivalent classes of Cauchy sequences in $\mathcal{S}$ with the above inner product the completion of $\mathcal{S}$ and denote it by $L^2(\mathbb{R})$.*

Remark: Anther way to look at $L^2(\mathbb{R})$ is to use the *Lebesgue integral* theory (see Appendix 2), which gives the following isomorphism

$$L^2(\mathbb{R}) \simeq \{f \in \mathcal{M}(\mathbb{R}) : \int_{-\infty}^{\infty} |f|^2 dx < \infty\}/\sim,$$

where $\mathcal{M}(\mathbb{R})$ denote the space of Lebesgue measurable complex valued functions on $\mathbb{R}$ and

$$f \sim g \Leftrightarrow f = g \;\; a.e. \text{ on } \mathbb{R}.$$

The Plancherel identity implies that both the Fourier transform $\hat{f}$ and the Fourier inversion $(\hat{f})^\vee$ extend to $L^2(\mathbb{R})$ on which the Fourier inversion formula and the Plancherel identity still hold. See *Exercise set 4* for a canonical basis of $L^2(\mathbb{R})$ using eigenfunctions of the Fourier transform.

2.4. **Fourier transform of tempered distributions.** In applications (e g elementary solution or Green's function of a partial differential operator, see Theorem 7.1.20 in [8] and the Week 5-2 exercise 5), it is crucial to extend Fourier transforms to a larger class of functions. A natural way to do it is to use the notion of *distribution* introduced by Schwartz.

Test functions: Denote by $\mathcal{D}_R$ the space of smooth functions, say f, on $\mathbb{R}$ such that

$$f(x) = 0, \quad \text{if } |x| \geqslant R.$$

Put

$$\mathcal{D} = \cup_{R>0}\mathcal{D}_R.$$

We call $\mathcal{D}$ the space of *test functions*. It is clear that $\mathcal{D}$ is a subspace of $\mathcal{S}$.

Exercise: $\mathcal{D}$ is not empty. Check that the classical cut-off function (see Wikipedia for "Mollifier")

$$\chi(x) := ce^{\frac{1}{|x|^2-1}}, \;\; |x| < 1; \quad \chi(x) := 0, \;\; |x| \geqslant 1, \tag{2.6}$$

is smooth on $\mathbb{R}$, where c is choosing such that $\int_{\mathbb{R}} \chi\, dx = 1$. For $\varepsilon > 0$, put

$$\chi_\varepsilon(x) := \varepsilon^{-1}\chi(\varepsilon^{-1}x),$$

then $\chi_\varepsilon \in \mathcal{D}_\varepsilon$. Moreover, for every $\delta > \varepsilon$,

$$\chi_{\varepsilon,\delta}(x) := \int_{|y|\leqslant\delta} \chi_\varepsilon(x-y)\, dy$$

lies in $\mathcal{D}_{\varepsilon+\delta}$ and $\chi_{\varepsilon,\delta}(x) = 1$ if $|x| \leqslant \delta - \varepsilon$.

Definition 2.5. *A $\mathbb{C}$-linear map*

$$T : \mathcal{D} \to \mathbb{C}$$

is said to be a distribution on $\mathbb{R}$ if for every $R > 0$ there exists a positive constant $C(R)$ and a positive integer $N(R)$ such that

$$|T(f)| \leqslant C(R) \sup_{|x|<R,\, 0\leqslant n\leqslant N(R)} |f^{(n)}(x)|, \qquad \forall\ f \in \mathcal{D}_R.$$

A distribution T on $\mathbb{R}$ is said to be tempered if it extends to a $\mathbb{C}$-linear map, still denote it by T,

$$T : \mathcal{S} \to \mathbb{C},$$

such that there exists a positive constant C and a positive integer N with

$$|T(f)| \leqslant C||f||_N, \ \ ||f||_N := \sup_{x\in\mathbb{R},\, 0\leqslant k,l\leqslant N} |x|^k |f^{(l)}(x)|, \qquad \forall\ f \in \mathcal{S}.$$

We shall denote by $\mathcal{D}'$ the space of all distrubutions on $\mathbb{R}$ and by $\mathcal{S}'$ the space of all tempered distributions on $\mathbb{R}$.

Remark: Notice that (try!)

$$\lim_{n\to\infty} ||f - \chi_{1,n} f||_N \to 0 = 0, \qquad \forall\ f \in \mathcal{S}$$

implies that every tempered distribution is uniquely determined by its restriction on $\mathcal{D}$.

Examples of distribution:

Piecewise continuous functions: If f is a piecewise continuous function then

$$T_f : g \mapsto \int_{\mathbb{R}} f(x)g(x)\, dx,$$

defines a distribution (sometimes we identity f with T_f). Check that T_{e^x} is not tempered.

Dirac's delta function: Fix $\xi \in \mathbb{R}$, the Delta funcion

$$\delta_\xi : f \mapsto f(\xi),$$

defines a distribution on $\mathbb{R}$, moreover $\delta_\xi \in \mathcal{S}'$

L^2-functions: One may look at $L^2(\mathbb{R})$ as a subspace of $\mathcal{S}'$: in fact, for every $f = [\{f_n\}]$,

$$T_f : g \mapsto \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)g(x)\, dx, \qquad g \in \mathcal{S},$$

defines a tempered distribution (we will identify f with T_f) since

$$\Big| \int_{\mathbb{R}} f_n(x)g(x)\, dx \Big| \leqslant \int_{\mathbb{R}} |f_n(x)|(1+x^2)^{-\frac{1}{2}}\, dx \cdot \sup_{x\in\mathbb{R}} |(1+x^2)^{\frac{1}{2}}|g(x)|$$

and

$$\int_{\mathbb{R}} |f_n(x)|(1+x^2)^{-\frac{1}{2}}\, dx \leqslant ||f_n|| \left(\int_{\mathbb{R}} (1+x^2)^{-1}\, dx \right)^{\frac{1}{2}} = \pi^{\frac{1}{2}} \cdot ||f_n||$$

give

$$|T_f(g)| \leqslant (2\pi)^{\frac{1}{2}} \cdot ||f|| \sup_{x\in\mathbb{R},\ 0\leqslant k\leqslant 1} |x|^k |g(x)|.$$

Cauchy principal values: The Cauchy principal value

$$p.v.\left(\frac{1}{x}\right) : f \mapsto \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{f(x)}{x}\, dx$$

defines a tempered distribution. In fact, we have

$$\int_{|x|>\varepsilon} \frac{f(x)}{x}\, dx = \int_{\varepsilon}^{\infty} \frac{f(x)-f(-x)}{x}\, dx.$$

Since

$$\Big|\frac{f(x)-f(-x)}{x}\Big| = \Big|\frac{1}{x}\int_{-x}^{x} f'(t)\, dt\Big| \leqslant 2\sup_{|t|\leqslant x} |f'(t)|, \quad x>0,$$

we get

$$\Big|\int_{\varepsilon}^{1} \frac{f(x)-f(-x)}{x}\, dx\Big| \leqslant 2\sup_{x\in\mathbb{R}} |f'(x)|,$$

and

$$\Big|\int_{1}^{\infty} \frac{f(x)-f(-x)}{x}\, dx\Big| \leqslant 2\left(\sup_{x\in\mathbb{R}} |xf(x)|\right)\int_{1}^{\infty} \frac{1}{x^2}\, dx = 2\sup_{x\in\mathbb{R}} |xf(x)|.$$

Thus

$$\Big|\Big[p.v.\left(\frac{1}{x}\right)\Big](f)\Big| \leqslant 2\sup_{x\in\mathbb{R}} |f'(x)| + 2\sup_{x\in\mathbb{R}} |xf(x)|,$$

which implies that $p.v.\left(\frac{1}{x}\right)$ defines a tempered distribution.

Derivative of a distribution. Let $T \in \mathcal{D}'$. The derivatives of T are always well defined

$$T^{(k)} : f \mapsto T((-1)^k f^{(k)}).$$

Notice that $T \in \mathcal{S}'$ implies that $T^{(k)} \in \mathcal{S}'$. If g is a smooth function then $T_g^{(k)} = T_{g^{(k)}}$.

Multiplication by smooth functions: Let $f \in \mathcal{D}, T \in \mathcal{D}'$. Then

$$fT(g) := T(fg),$$

defines $fT \in \mathcal{D}'$. It is easy to check that

$$xT \in \mathcal{S}', \quad fT \in S',$$

if $f \in \mathcal{S}, T \in \mathcal{S}'$.

2.4.1. *Fourier transform of a tempered distribution.* The motivation for extending Fourier transform to tempered distributions comes from the following identity (which is sometimes called the multiplication formula)

Theorem 2.4. *If $f, g \in \mathcal{S}$ then*

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)\, dx = \int_{-\infty}^{\infty} \hat{f}(y)g(y)\, dy.$$

Proof. Put $F(x,y) = f(x)g(y)e^{-2\pi ixy}$ then the theorem follows from

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} F(x,y)\, dy\right) dx = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} F(x,y)\, dx\right) dy,$$

the details are left to the readers. □

Definition 2.6 (Fourier transform on $\mathcal{S}'$). *Let $T \in \mathcal{S}'$, put*

$$\hat{T} : f \mapsto T(\hat{f}), \quad \check{T} : f \mapsto T(\check{f}),$$

we call $\hat{T}$ the Fourier transform of T and $\check{T}$ the inverse Fourier transform of T.

Remark: The following lemma implies that $\hat{T}, \check{T} \in \mathcal{S}'$ if $T \in \mathcal{S}'$.

Lemma 2.2. *For every $f \in \mathcal{S}$, we have*

$$||\hat{f}||_N \leqslant C(N)||f||_{N+2}$$

Proof. Follows from

$$|x^k \hat{f}^{(l)}(x)| = |\widehat{(y^l f)^{(k)}}(x)| \leqslant \int_{\mathbb{R}} |(y^l f)^{(k)}|\, dy \leqslant \pi \sup_{y\in\mathbb{R}}(1+y^2)|(y^l f)^{(k)}|.$$

□

One may define $f \star T$ ($f \in \mathcal{S}, T \in \mathcal{S}'$) as follows

$$(f \star T)(h) := T(f^- \star h), \qquad h \in \mathcal{S},$$

where $f^-(x) := f(-x)$. Then the following basic properties of the Fourier transform can be naturally generalized to tempered distributions.

Theorem 2.5. *For every $f, g \in \mathcal{S}$, $T \in \mathcal{S}'$, we have*

1) $\widehat{f \star T} = \hat{f}\hat{T}$;
2) $\widehat{T^{(k)}} = (2\pi ix)^k\, \hat{T}$;
3) $\widehat{-2\pi ixT} = (\hat{T})^{(1)}$;
4) $\check{\hat{T}} = T$.

Example: Fourier transform of the Delta function δ_ξ:

$$\widehat{\delta_\xi}(f) = \delta_\xi(\hat{f}) = \hat{f}(\xi) = T_{e^{-2\pi i x\xi}}(f).$$

Thus

$$\widehat{\delta_\xi} = T_{e^{-2\pi i x\xi}}$$

and the Fourier inversion formula gives

$$\widehat{T_{e^{2\pi i x\xi}}} = \delta_\xi.$$

Sometimes we just write $\widehat{\delta_\xi} = e^{-2\pi i x\xi}$ and $\widehat{e^{2\pi i x\xi}} = \delta_\xi$.

2.4.2. *Poisson summation formula and periodic distributions.* We shall follow page 177–181 in [8]. By Lemma 2.1, we know that

$$u := \sum_{k\in\mathbb{Z}} \delta_k,$$

defines a tempered distribution. From the definition, we know that the Poisson summation formula is equivalent to

Theorem 2.6. *The Fourier transform of u is equal to itself.*

Definition 2.7. *A distribution $T \in \mathcal{D}'$ is said to be 1-periodic if*

$$T(f) = T(f_n), \ \ \forall\, f \in \mathcal{D},\ n \in \mathbb{Z},$$

where $f_n(x) := f(x+n)$.

It is clear that u is 1-periodic. In general, let $T \in \mathcal{D}'$ be 1-periodic, put

$$\phi_n(x) = \frac{\chi(x+n)}{\sum_{k\in\mathbb{Z}} \chi(x+k)},$$

where χ is defined in (2.6), then $\phi_0 \in \mathcal{D}_1$ and $\sum_{n\in\mathbb{Z}} \phi_n = 1$ gives

$$T(f) = \sum_{n\in\mathbb{Z}} T(f\phi_n) = \sum_{n\in\mathbb{Z}} T(f_n\phi_0), \ \ \forall\, f \in \mathcal{D}.$$

The above formula also implies that *every* 1-*periodic distribution is tempered*. Together with the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} \hat{f}(x+n) = \sum_{n\in\mathbb{Z}} f(n) e^{-2\pi i n x}$$

it gives

$$T(\hat{f}) = \sum_{n\in\mathbb{Z}} T(e^{-2\pi i n x}\phi_0) f(n).$$

Thus we get

Theorem 2.7. *If $T \in \mathcal{D}'$ is 1-periodic then*

$$\hat{T} = \sum_{n\in\mathbb{Z}} T_{\mathbb{R}/\mathbb{Z}}(e^{-2\pi inx})\delta_n,$$

where $T_{\mathbb{R}/\mathbb{Z}}(e^{-2\pi inx}) := T(e^{-2\pi inx}\phi_0)$ does not depend on ϕ_0.

Remark: In case $T = T_g$ for a 1-periodic continuous function g, we have

$$T(e^{-2\pi inx}\phi_0) = \int_0^1 g(x)e^{-2\pi inx}\,dx = \hat{g}(n)$$

Moreover, by the Fourier inversion formula, the above theorem gives

$$\int_{\mathbb{R}} f(x)g(x)\,dx = T(f) = \hat{T}(\widehat{f^-}) = \sum_{n\in\mathbb{Z}} \hat{g}(n)\int_{\mathbb{R}} f(x)e^{2\pi inx}\,dx,$$

which can be seen as a generalization of the Fourier series expansion $g(x) \sim \sum_{n\in\mathbb{Z}} \hat{g}(n)e^{2\pi inx}$.

2.4.3. *Elementary solution of $D + s$.* Recall that the Dirac operator D is defined by $-i\frac{d}{dx}$.

Definition 2.8. *Fix $\xi \in \mathbb{R}$, $s \in \mathbb{C}\backslash 2\pi\mathbb{Z}$. If a 1-periodic distribution T satisfies*

$$(D + s)T = \sum_{k\in\mathbb{Z}} \delta_{\xi+k},$$

then we call T an elementary solution of $D + s$ on $\mathbb{R}/\mathbb{Z}$.

Assume that T is an elementary solution. Apply the Fourier transform to the equation, Theorem 2.7 gives

$$(2\pi x + s)\hat{T} = \sum_{n\in\mathbb{Z}} e^{-2\pi in\xi}\delta_n.$$

Since $s \in \mathbb{C}\backslash 2\pi\mathbb{Z}$, we get

$$\hat{T} = \sum_{n\in\mathbb{Z}} \frac{e^{-2\pi in\xi}}{2\pi n + s}\delta_n,$$

which gives uniqueness of T. Moreover, since $\check{\delta_n} = e^{2\pi inx}$, one may guess that T is given by the following function

$$f(x) = \sum_{n\in\mathbb{Z}} \frac{e^{2\pi in(x-\xi)}}{2\pi n + s}.$$

Recall that we have proved that

$$\frac{\pi}{\sin \pi s}e^{i(\pi-x)s} = \sum_{n\in\mathbb{Z}} \frac{e^{inx}}{n+s}, \text{ if } x \in (0, 2\pi), \quad \pi \cot \pi s = \sum_{n\in\mathbb{Z}} \frac{1}{n+s}.$$

in Theorem 1.6, from which we get

Theorem 2.8. *The elementary solution of $D + s$ on $\mathbb{R}/\mathbb{Z}$ is unique and can be written as T_f, where f is a piecewise smooth 1-period function. When $\xi \in (0,1)$, we have (see the remark below)*

$$f(x) = ie^{is(\xi - x)} \left(H(x-\xi) + \frac{1}{e^{is}-1} \right),$$

when $x \in [0,1]\backslash\{\xi\}$ and

$$f(\xi) = i\left(\frac{1}{2} + \frac{1}{e^{is}-1}\right) = \frac{1}{2}\cot(s/2).$$

Remark: Assume that $\xi \in (0,1)$. Another way to find the elementary solution is to find a piecewise smooth function on $[0,1]$ such that

$$-if'(x) + sf(x) = \delta_\xi, \quad f(0) = f(1),$$

where the first equation is defined in the sense of distribution. Let us rewrite the equation as

$$(fe^{isx})' = ie^{is\xi}\delta_\xi.$$

It is easy (try!) to check that the following *Heaviside function*

$$H(x) = 0, \ \ x < 0, \quad H(x) = 1, \ \ x > 1, \ \ H(0) = \frac{1}{2}$$

satisfies $H' = \delta_0$. Thus we have

$$fe^{isx} = ie^{is\xi}H(x-\xi) + C.$$

Now $f(0) = f(1)$ gives

$$C = \frac{ie^{is\xi}}{e^{is}-1}.$$

Thus we get

$$f(x) = ie^{is(\xi - x)} \left(H(x-\xi) + \frac{1}{e^{is}-1} \right),$$

when $x \in [0,1]\backslash\{\xi\}$ and

$$f(\xi) = i\left(\frac{1}{2} + \frac{1}{e^{is}-1}\right) = \frac{1}{2}\cot(s/2).$$

2.4.4. *Wave and heat equations associated to D.*

Definition 2.9. *Let D be a selfadjoint operator, we call $\partial/\partial t + D$ the heat operator associated to D and $\partial^2/\partial t^2 + D$ the wave operator associated to D.*

Remark 1: In case D is the Dirac operaror $-i\frac{d}{dx}$, then we can have

$$\partial/\partial t + D = \partial/\partial t - i\partial/\partial x.$$

Consider the complex coordinate

$$z := x + it, \quad \bar{z} := x - it$$

then we have

$$\partial/\partial\bar{z} = \frac{1}{2}\left(\partial/\partial x + i\partial/\partial t\right) = \frac{i}{2}\left(\partial/\partial t + D\right).$$

Thus up to a constant, the heat operator associated to D is equal to the *Cauchy–Riemann operator* $\partial/\partial\bar{z}$ whose kernel gives the holomorphic functions.

Remark 2: Still let D be the Dirac operaror $-i\frac{d}{dx}$, then the wave operator associated to D is

$$\partial^2/\partial t^2 - i\partial/\partial x,$$

this operaor can be seen as a time-dependent Schrödinger operator with trivial potential energy, see [15] for a nice introduction of the Schrödinger equation.

Remark 3: In case D is the Laplacian $-d^2/dx^2$, i e square of the Dirac operator, then

$$\partial/\partial t - \partial^2/\partial x^2$$

is just the classical heat operator and

$$\partial^2/\partial t^2 - \partial^2/\partial x^2$$

is the classical wave operator .

2.4.5. *Fourier transform of the Heaviside function.* Recall that

$$2\pi i x\widehat{H-C} = \widehat{H'} = \hat{\delta}_0 = 1,$$

thus one might guess that

$$\widehat{H-C} = \frac{1}{2\pi i}\, p.v.\left(\frac{1}{x}\right)$$

and C should be $\frac{1}{2}$ since the right hand side is odd.

Theorem 2.9. $\widehat{H-\frac{1}{2}} = \frac{1}{2\pi i}\, p.v.\left(\frac{1}{x}\right)$.

Proof. By definition, for every $f \in \mathcal{S}$, we have

$$\widehat{H-\frac{1}{2}}(f) = \frac{1}{2}\left(\int_0^\infty \hat{f}(x)\, dx - \int_{-\infty}^0 \hat{f}(x)\, dx\right) = \frac{1}{2}\lim_{k\to\infty}\int_{\frac{1}{k}}^{k} \hat{f}(x) - \hat{f}(-x)\, dx.$$

Definition of $\hat{f}$ and the Euler formula give

$$\frac{\hat{f}(x) - \hat{f}(-x)}{2} = -i\int_{\mathbb{R}} f(y)\sin(2\pi xy)\, dy.$$

Thus

$$\widehat{H-\frac{1}{2}}(f) = \frac{1}{-2\pi i}\lim_{k\to\infty}\int_{y\in\mathbb{R}} f(y)\frac{\cos(2\pi ky) - \cos(2\pi k^{-1}y)}{y}\, dy,$$

Since $y^{-1}\cos ay$ is odd, we have

$$\int_{y\in\mathbb{R}} f(y)\frac{\cos(2\pi ky) - \cos(2\pi k^{-1}y)}{y}\, dy = \int_0^\infty (f(y) - f(-y))\frac{\cos(2\pi ky) - \cos(2\pi k^{-1}y)}{y}\, dy.$$

Since $f \in \mathcal{S}$, we have

$$|\int_N^\infty (f(y) - f(-y)) \frac{\cos(2\pi k y) - \cos(2\pi k^{-1} y)}{y}\, dy| \leqslant N^{-1},$$

when N is large enough. Moreover, the Riemann-Lebesgue lemma implies that

$$\lim_{k\to\infty} \int_0^N \frac{f(y) - f(-y)}{y} \cos(2\pi k y)\, dy = 0$$

since $\frac{f(y)-f(-y)}{y}$ is continuous on $[0, N]$. Thus

$$\widehat{H - \frac{1}{2}}(f) = \frac{1}{2\pi i} \lim_{N\to\infty} \lim_{k\to\infty} \int_0^N \frac{f(y) - f(-y)}{y} \cos(2\pi k^{-1} y)\, dy = \frac{1}{2\pi i} \int_0^\infty \frac{f(y) - f(-y)}{y}\, dy.$$

Thus the theorem follows. □

2.5. **Fourier–Laplace transform and Paley–Wiener theorem.** The aim is study the Fourier–Laplace transform and prove the Paley–Wiener theorem with the help of Cauchy integral formula and the maximum principle in complex analysis. We shall follow Chapter 1 of Hörmander's book [9] in this section. Our starting point is the following basic observation: if $g \in \mathcal{D}_R$ then

$$\hat{g}(z) = \int_{\mathbb{R}} g(y) e^{-2\pi i y z}\, dy = \int_{-R}^{R} g(y) e^{-2\pi i y z}\, dy,$$

defines a holomorphic function on $z \in \mathbb{C}$, i.e. $\hat{g}$ is an *entire function*.

Definition 2.10. *We call $\hat{g}(z)$, $z \in \mathbb{C}$ the Fourier–Laplace transform of $g \in \mathcal{D}_R$.*

Let us write

$$z = a + ib, \qquad a, b \in \mathbb{R}$$

then

$$\hat{g}(a + ib) = \int_{-R}^{R} g(y) e^{-2\pi i a y} e^{2\pi b y}\, dy$$

gives

$$|\hat{g}(a + ib)| \leqslant e^{2\pi R|b|} \int_{-R}^{R} |g(y)|\, dy.$$

Moreover, for every natural number k, integration by parts gives

$$\widehat{g^{(k)}} = (2\pi i z)^k \hat{g}(z).$$

Thus

$$(2\pi)^k |z^k \hat{g}(z)| \leqslant e^{2\pi R|b|} \int_{-R}^{R} |g^{(k)}(y)|\, dy.$$

The smooth version of the Paley–Wiener theorem can be seen as an *inverse statement* of the above estimate.

Theorem 2.10 (Paley–Wiener–Schwartz). *Let U be an entire function such that*

$$(1+|z|)^k\,|U(z)| \leqslant C_k\, e^{2\pi R\,|\mathrm{Im}\, z|}, \tag{2.7}$$

for every non-negative integer k. Then there exists $u \in \mathcal{D}_R$ whose Fourier–Laplace transform is U.

Proof. By the Fourier inversion formula, it suffices to define

$$u(x) = \int_{\mathbb{R}} U(y) e^{2\pi i x y}\, dy \tag{2.8}$$

and check that $u \in \mathcal{D}_R$. By (2.7), we know that $U|_{\mathbb{R}} \in \mathcal{S}$, thus $u \in \mathcal{S}$. Now it is enough to check that $u(x) = 0$ if $|x| > R$. The key of the proof is to use the Cauchy integral formula: notice that (2.7) permits us to shift the integration in (2.8) into the complex domains, which gives

$$u(x) = \int_{\mathbb{R}} U(y+ib) e^{2\pi i x (y+ib)}\, dy,$$

for every $b \in \mathbb{R}$. Estimating the integral by means of (2.7) with $k=2$, we obtain

$$|u(x)| \leqslant C_2 e^{-2\pi x b + 2\pi R b} \int_{y \in \mathbb{R}} (1+|y|)^{-2}\, dy,$$

and the integral is convergent. If we choose $b = tx$ and let $t \to \infty$, it now follows that $u(x) = 0$ if $|x| > R$. □

In the next section, we shall study a distribution version of the above theorem.

2.5.1. *Fourier–Laplace transform of distributions with bounded support.*

Definition 2.11. *The support of a complex function, f, on $\mathbb{R}$ is defined as*

$$\mathrm{Supp}\, f := \{x \in \mathbb{R} : f(x) \neq 0\}.$$

Let K be a bounded subset in $\mathbb{R}$. We shall denote by $\mathcal{E}(K)$ the space of smooth functions on $\mathbb{R}$ with support in K.

Definition 2.12. *Let T be a distribution on $\mathbb{R}$. Fix $x \in \mathbb{R}$, we say that T is zero at x if there exists $\varepsilon > 0$ such that*

$$T(f) = 0, \qquad \forall\, f \in \mathcal{E}([x-\varepsilon, x+\varepsilon]).$$

The support $\mathrm{Supp}\, T$ of T is defined as the set of points where T is not zero. We say that T has compact support if $\mathrm{Supp}\, T$ is bounded. Let K be a bounded subset in $\mathbb{R}$. We shall denote by $\mathcal{E}'(K)$ the space of distributions with support in K.

Example: Let g be a continuous function on $\mathbb{R}$ with support in $[-R, R]$. Then for each $k \geqslant 1$, the k-th order derivative, $T_g^{(k)}$, of T_g lies in $\mathcal{E}'([-R, R])$. In fact, every distribution on $\mathcal{E}'([-R, R])$ can be represented in this way (see [9] for related results). In particular, every distribution with bounded support is tempered.

Remark: One may prove that (try! or see page 11 in [9]) a distribution T lies in $\mathcal{E}'(K)$ if and only if there exist constant C and k such that

$$|T(f)| \leqslant C \sum_{x\in K,\, 0\leqslant n\leqslant k} |f^{(n)}(x)|, \qquad \forall\, f \in C^\infty(\mathbb{R}).$$

In particular, for every $z \in \mathbb{C}$, $y \mapsto e^{-2\pi i y z}$ is smooth, thus the following function

$$\hat{T} : z \mapsto T(e^{-2\pi i y z})$$

is well defined in case the distribution T has a bounded support.

Definition 2.13. *We call $\hat{T}$ the Fourier–Laplace transform of T.*

Remark: By the fact in the **Example** above, we know that (try!) if $T \in \mathcal{E}'([-R, R])$ then $\hat{T}$ is a holomorphic function on $\mathbb{C}$ such that

$$(1 + |z|)^{-k}\, |\hat{T}(z)| \leqslant C_k\, e^{2\pi R\, |\mathrm{Im}\, z|},$$

for some integer k and constant C_k. The *inverse statement* is the following distribution version of Theorem 2.10.

Theorem 2.11 (Distribution version of the Paley–Wiener theorem)**.** *Let U be an holomorphic function on $\mathbb{C}$ such that*

$$(1 + |z|)^{-k}\, |U(z)| \leqslant C_k\, e^{2\pi R\, |\mathrm{Im}\, z|}, \tag{2.9}$$

for some integer k. Then there exists a unique $T \in \mathcal{E}'([-R, R])$ such that $\hat{T} = U$.

Proof. First note that $U \in \mathcal{S}'$, thus we can write $U = \hat{T}$ for some $T \in \mathcal{S}'$. Then $\widehat{\chi_\varepsilon \star T} = \hat{\chi}_\varepsilon \hat{T}$ satisfies Theorem 2.10 with R replaced by $R+\varepsilon$. Thus each $\mathrm{Supp}\,(\chi_\varepsilon \star T)$ lies in $[-R-\varepsilon, R+\varepsilon]$ and when ε goes to zero this implies that $T \in \mathcal{E}'([-R, R])$. □

The above theorem gives directly the following weak L^2-version of the Paley–Wiener theorem.

Theorem 2.12. *A holomorphic function U on $\mathbb{C}$ is the Fourier–Laplace transform of an L^2 function on $[-R, R]$ if and only if*

$$(1 + |z|)^{-k}\, |U(z)| \leqslant C_k\, e^{2\pi R\, |\mathrm{Im}\, z|},$$

for some integer k and the L^2-norm of U on the real line is finite.

Remark: If we look at $L^2[-R, R]$ as the completion of $\mathcal{D}$ with respect to the following norm

$$||f||^2 := \int_{-R}^{R} |f(x)|^2\, dx, \quad f \in \mathcal{D},$$

then Fourier–Laplace transform of $f = [\{f_n\}] \in L^2[-R, R]$ can be defined as

$$\hat{f}(z) := \lim_{n\to\infty} \int_{-R}^{R} f_n(x) e^{-2\pi i x z}\, dx.$$

The classical L^2-version of the Paley–Wiener theorem will be proved in the next section.

2.5.2. *Classical L^2-version of the Paley–Wiener theorem.* We shall use a version of Phragmén–Lindelöf Theorem in page 108–109 in [14].

Definition 2.14. *We say that a A holomorphic function F on $\mathbb{C}$ is of exponential type $T > 0$ if for every $\varepsilon > 0$ there exists a constant A_ε such that*

$$|F(z)| \leqslant A_\varepsilon e^{(T+\varepsilon)|z|},$$

on $\mathbb{C}$.

Theorem 2.13 (Paley–Wiener theorem)**.** *A holomorphic function U on $\mathbb{C}$ is the Fourier–Laplace transform of an L^2 function on $[-R, R]$ if and only if U is of exponential type $2\pi R$ and*

$$\int_{\mathbb{R}} |U(x)|^2 \, dx < \infty.$$

We will prove the following stronger version the Paley–Wiener theorem.

Theorem 2.14 (Strong Paley–Wiener theorem)**.** *Let U be a holomorphic function on $\mathbb{C}$ whose restriction to the real line is L^2. Then the followings are equivalent:*

1) *U is of exponential type $2\pi R$;*
2) *For all $y \in \mathbb{R}$, we have*
$$\int_{\mathbb{R}} |U(x+iy)|^2 \, dx \leqslant e^{4\pi R|y|} \int_{\mathbb{R}} |U(x)|^2 \, dx;$$
3) *For all $\varepsilon > 0$, we have*
$$|U(z)|^2 \leqslant \frac{2e^{4\pi R(|\mathrm{Im}\, z|+\varepsilon)}}{\pi\varepsilon} \int_{\mathbb{R}} |U(x)|^2 \, dx;$$
4) *U is the Fourier–Laplace transform of an L^2 function on $[-R, R]$;*
5) *$(1+|z|)^{-k} \, |U(z)| \leqslant C_k \, e^{2\pi R\, |\mathrm{Im}\, z|}$ for some integer k.*

Proof. It is clear that $5)$ implies $1)$. The fact that $2)$ implies $3)$ follows from the sub-mean inequality (try to prove it using taylor expansion at z_0)

$$|U(z_0)|^2 \leqslant \frac{1}{\pi\varepsilon^2} \int_{|z-z_0|<\varepsilon} |U(z)|^2 \, dxdy.$$

For $3)$ implies $4)$, by the weak L^2-version of of the Paley–Wiener theorem, we know that U is the Fourier Laplace transform of an L^2 function, say u, on $[-R-\varepsilon, R+\varepsilon]$ for every $\varepsilon > 0$. Since u is uniquely determined by U, we know that the support of u lies in $[-R, R]$. Now the weak L^2-version of the Paley–Wiener theorem gives that $4)$ implies $5)$. Now it suffices to show that $1)$ implies $2)$ (will be proved at the end of this section). The key is the following variant of maximum principle (so called the *Phragmén–Lindelöf Theorem*). □

Theorem 2.15 (Phragmén–Lindelöf Theorem). *Assume that f is holomorphic and $|f(z)| \leqslant Ae^{|z|^\beta}$ in a sector D of (angular) opening less than π/α. If f is also continuous in the closed section $\bar{D}$ and $0 \leqslant \beta < \alpha$ then*

$$\sup_{z\in\bar{D}} |f(z)| = \sup_{z\in\partial D} |f(x)|$$

Proof. Assume that $\sup_{z\in\partial D} |f(x)| = M$. By a rotation, one may assume that

$$D = \{re^{i\theta} : r > 0,\ |\theta| < \psi\},$$

where $\psi < \frac{\pi}{2\alpha}$. Pick $\beta < \gamma < \alpha$ and define

$$F(z) = f(z)e^{-\varepsilon z^\gamma},$$

for every $\varepsilon > 0$. It follows that $|F(z)| \leqslant |f(z)| \leqslant M$ on the two bounding rays. Moreover, on the arc $z = Re^{i\theta}|$, $|\theta| \leqslant \frac{\pi}{2\alpha}$,

$$|F(z)| \leqslant Ae^{R^\beta - \varepsilon R^\gamma \cos(\gamma\pi/2\alpha)},$$

which tends to zero as R goes to infinity. Thus $|F(z)| \leqslant \frac{1}{N}$ for every N proved R is large enough. The maximum principle, thus, implies that

$$|F(z)| \leqslant \lim_{N\to\infty} \max\{1/N, M\} = M$$

on $\bar{D}$, which gives that

$$|f(z)| \leqslant Me^{\varepsilon r^\gamma \cos(\gamma\theta)}$$

for all $z = re^{i\theta} \in \bar{S}$, Letting $\varepsilon \to 0$ we obtain our result. □

Lemma 2.3. *Suppose F is of exponential type T and $|F(x)| \leqslant 1$ for x real then $|F(x+iy)| \leqslant e^{T|y|}$ for all complex numbers $z = x + iy$.*

Proof. For $\varepsilon > 0$ set

$$F_\varepsilon(z) = F(z)e^{i(T+\varepsilon)z}.$$

Since F is of exponential type T

$$|F_\varepsilon(iy)| = |F(iy)|e^{-(T+\varepsilon)y} \leqslant A_\varepsilon,$$

for all non-negative y. We also have $|F_\varepsilon(x)| \leqslant 1$ for all real x. Thus gives us a bound for F on the positive x and y exes. Moreover, we certainly can find B so that

$$|F_\varepsilon(z)| \leqslant A_\varepsilon e^{(T+\varepsilon)(|z|-y)} \leqslant A_\varepsilon e^{2(T+\varepsilon)|z|} \leqslant Be^{|z|^{\frac{3}{2}}}.$$

We can therefore apply the Phdagmén–Lindelöf theorem with $\beta = \frac{3}{2} < 2 = \alpha$ and obtain

$$|F_\varepsilon(z)| \leqslant \max\{A_\varepsilon, 1\} := A$$

for all $z = x + iy$ such that $x \geqslant 0$ and $y \geqslant 0$. If we now repeat this argument for the second quadrant, we can apply the Phdagmén–Lindelöf theorem again to F_ε on the upper half-plane and $\beta = 0 < 1 = \alpha$ to obtain $|F(x+iy) \leqslant 1$ for $y \geqslant 0$. Letting $\varepsilon \to 0$ we obtain $|F(x+iy)| \leqslant e^{Ty}$ for $y \geqslant 0$. The lemma is then established by applying the result to $G(z) = F(-z)$. □

Proof of 1) *implies* 2). The main idea is to use the following identity

$$\int_{\mathbb{R}} |U(x+iy)|^2\,dx = \sup |G(iy)|^2, \quad G(z) := \int_{\mathbb{R}} U(z+t)f(t)\,dt,$$

where the supremum is taken over all smooth functions f with bounded support in $\mathbb{R}$ such that $\int_{\mathbb{R}} |f(t)|^2\,dt = 1$. By the Schwartz inequality, we know that

$$|G(x)|^2 \leqslant \int_{\mathbb{R}} |U(x+t)|^2 dt \cdot \int_{\mathbb{R}} |f(t)|^2\,dt = \int_{\mathbb{R}} |U(t)|^2 dt := A^2$$

Apply the above lemma to $F(z) := G(z)/A$ (notice that 1) implies that F is of exponential type $2\pi R$), we get

$$|G(x+iy)| \leqslant e^{2\pi R|y|}A.$$

Thus 2) follows. □

Remark: For other proofs and related results, see Seip's notes [11] or page 158–160 in [3]).

2.6. **Sampling theorem.** The Paley–Wiener theorem tells us that one can hear the support (bandwidth) of a signal by estimating the exponential order of its Fourier–Laplace transform. In this section, we shall study a sampling property of Fourier–Laplace transform of an L^2-function with bounded support. The main result is the following,, see page 167 in [12].

Theorem 2.16. *Let $f \in L^2[-1/2, 1/2]$. Identify f with a function on $\mathbb{R}$ supported on $[-1/2, 1/2]$. Then for every $x \in \mathbb{R}$, we have*

$$\hat{f}(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) \frac{\sin \pi(x-n)}{\pi(x-n)},$$

and

$$\int_{\mathbb{R}} |\hat{f}(x)|^2\,dx = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2.$$

Proof. Since $\{e^{2\pi inx}\}$ is an orthonormal basis of $L^2[-1/2, 1/2]$, we know that f is the L^2 limit of the following sequence of functions

$$f_N := \sum_{|n|\leqslant N} (f, e^{2\pi inx}) e^{2\pi iny},$$

as N goes to infinity. Notice that

$$(f, e^{2\pi inx}) = \int_{|x|<1/2} f(x)e^{-2\pi inx}\,dx = \int_{\mathbb{R}} f(x)e^{-2\pi inx}\,dx = \hat{f}(n).$$

Identity each f_N as a function on $\mathbb{R}$ supported on $[-1/2, 1/2]$, we know that the limit of f_N is f in $L^2(\mathbb{R})$. Since Fourier transform preserves the $L^2(\mathbb{R})$ norm (Plancherel identity) we know that $\hat{f}$ is equal to the L^2 limit of

$$\widehat{f_N}(x) = \sum_{|n|\leqslant N} \hat{f}(n) \int_{|y|<1/2} e^{-2\pi ixy} e^{2\pi iny}\,dy.$$

Now integration by parts gives

$$\int_{|y|<1/2} e^{-2\pi ixy} e^{2\pi iny}\, dy = \frac{\sin \pi(x-n)}{\pi(x-n)},$$

where we define the value at n of the right hand as 1. Thus the first formula follows. The second follows from

$$\int_{\mathbb{R}} |\hat{f}(x)|^2\, dx = \int_{|x|<1/2} |f(x)|^2\, dx = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2.$$

□

Remark 1: If we first look at the Fourier series expansion of f in $L^2[-\lambda/2, \lambda/2]$, $\lambda > 1$, then for an arbitray continuous function χ such that $\chi = 1$ on $[-1/2, 1/2]$ and $\chi = 0$ outside $[-\lambda/2, \lambda/2]$, we have

$$||f - \chi f_N|| = ||\chi f - \chi f_N|| \to 0, \quad N \to \infty,$$

which gives

$$||\hat{f} - \widehat{\chi f_N}|| \to 0, \quad N \to \infty,$$

in particular, if we choose χ such that $\chi(x)$ is linear when $1/2 < |x| < \lambda/2$, we will get the following oversampling formula

$$\hat{f}(x) = \sum_{n\in\mathbb{Z}} \frac{1}{\lambda} f\left(\frac{n}{\lambda}\right) K_\lambda\left(x - \frac{n}{\lambda}\right), \quad K_\lambda(y) := \frac{\cos \pi y - \cos \pi\lambda y}{\pi^2(\lambda-1)y^2}.$$

Thus, if one samples $\hat{f}$ "more often", the series in the above formula converges faster since $K_\lambda(y) = O(1/|y|^2)$ as $|y| \to \infty$.

Remark 2: The sampling theroem is essentially the Fourier tranform of the Fourier series expansion. In fact, since the Fourier transform preserves the L^2 inner product on $\mathbb{R}$, and

$$\{e^{2\pi inx}|_{|x|<1/2}\}$$

is an orthonormal system in $L^2(\mathbb{R})$, we know that it is Fourier transform

$$\{\frac{\sin \pi(x-n)}{\pi(x-n)}\}$$

is also an orthonormal system in $L^2(\mathbb{R})$. Thus the samping formula is also an orthogonal decomposition formula.

2.7. **Heisenberg uncertainty principle.** See page 158–161 and 168–169 in [12].

The mathematical thrust of the principle can be formulated in terms of a relation between a function and its Fourier transform. The basic underlying law, formulated in its vaguest and most general form, states that *a function and its Fourier transform cannot both be essentially localized*. Somewhat more precisely, if the "preponderance" of the mass of a func-tion is concentrated in an interval of length L, then the preponderance of the mass of its Fourier transform cannot lie in an interval of length essentially smaller than L^{-1}. The exact statement is as follows.

Theorem 2.17. *Assume that $f \in \mathcal{S}$ satisfies the normalization condition $\int_{\mathbb{R}} |f(x)|^2\,dx = 1$. Then*

$$\left(\int_{\mathbb{R}} x^2 |f(x)|^2\,dx\right) \cdot \left(\int_{\mathbb{R}} y^2 |\hat{f}(y)|^2\,dy\right) \geqslant \frac{1}{16\pi^2},$$

and equality holds iff $f(x) = Ae^{-Bx^2}$ where $B > 0$ and $A^2 = \sqrt{2B/\pi}$.

Proof. Step 1: Beginning with our normalizing assumption $\int_{\mathbb{R}} |f(x)|^2\,dx = 1$, and recalling that both f and f' are rapidly decreasing, an integration by parts gives

$$1 = \int_{\mathbb{R}} |f|^2\,dx = -\int_{\mathbb{R}} x\,(|f|^2)'\,dx = -\int_{\mathbb{R}} x\,(f'\bar{f} + f\bar{f}')\,dx.$$

Therefore

$$1 \leqslant 2\int_{\mathbb{R}} |x| \cdot |f| \cdot |f'|\,dx \leqslant 2||xf|| \cdot ||f'||,$$

where we have used the Cauchy–Schwartz inequality. The identities

$$||f'||^2 = ||\hat{f}'||^2 = ||2\pi i y \hat{f}||^2 = 4\pi^2 ||y\hat{f}||^2,$$

which hold because of the properties of the Fourier transform and the Plancherel formula, conlcudes the proof of the inequality in the theorem.

Step 2: If equality holds then we must also have equality where we applied the Cauchy–Schwarz inequality, and as a result we find that $f'(x) = cxf(x)$ for some constant c. The solutions to this equation are $f(x) = Ae^{cx^2/2}$, where A is constant. Since we want $f \in \mathcal{S}$, we must take $c = -2B < 0$, now $||f|| = 1$ gives $|A^2| = \sqrt{2B/\pi}$. □

Remark: Replacing f by $e^{-2\pi i x y_0} f(x + x_0)$ and changing variables one only get

$$\left(\int_{\mathbb{R}} (x - x_0)^2 |f(x)|^2\,dx\right) \cdot \left(\int_{\mathbb{R}} (y - y_0)^2 |\hat{f}(y)|^2\,dy\right) \geqslant \frac{1}{16\pi^2}$$

for every $x_0, y_0 \in \mathbb{R}$. The precise assertion contained in the above inequality first came to light in the study of *quantum mechanics*. It arose when one considered the extent to which one could simultaneously locate the position and momentum of a particle. Assuming we are dealing with (say) an electron that travels along the real line, then according to the laws of physics, matters are governed by a "state function" f, which we can assume to be in $\mathcal{S}$, and which is normalized according to the requirement that $||f|| = 1$. The position of the particle is then determined not as a definite point x; instead its probable location is given by the rules of quantum mechanics as follows:

The probability that the particle is located in the interval (a, b) is $\int_a^b |f(x)|^2\,dx$.

According to this law we can calculate the probable location of the particle with the aid of f: in fact, there may be only a small probability that the particle is located in a given interval (a', b'), but nevertheless it is somewhere on the real line since $\int_{\mathbb{R}} |f(x)|^2\,dx = 1$.

In addition to the *probability density* $|f(x)|^2\,dx$, there is the *expectation* of where the particle might be. This expectation is the best guess of the position of the particle, given its probability distribution determined by $|f(x)|^2\,dx$, and is the quantity defined by

$$E(x) := \int_{\mathbb{R}} x|f(x)|^2\,dx. \tag{2.10}$$

Why is this our best guess? Consider the simpler (idealized) situation where we are given that the particle can be found at only finitely many different points, $x_1, x_2, \cdots, x_N$ on the real axis, with p_i the probability that the particle is at x_i, and $p_1 + p_2 + \cdots + p_N = 1$. Then, if we knew nothing else, and were forced to make one choice as to the position of the particle, we would naturally take $E(x) = \sum x_i p_i$, which is the appropriate weighted average of the possible positions. The quantity (2.10) is clearly the general (integral) version of this.

We next come to the notion of *variance*, which in our terminology is the uncertainty attached to our expectation. Having determined that the expected position of the particle is $E(x)$ (given by (2.10)), the resulting uncertainty is the quantity

$$\int_{\mathbb{R}} (x - E(x))^2 |f(x)|^2\,dx. \tag{2.11}$$

Notice that if f is highly concentrated near $E(x)$, it means that there is a high probability that x is near $E(x)$, and so (2.11) is small, because most of the contribution to the integral takes place for values of x near $E(x)$. Here we have a small uncertainty. On the other hand, if $f(x)$ is rather flat (that is, the probability distribution $|f(x)|^2\,dx$ is not very concentrated), then the integral (2.11) is rather big, because large values of $(x - E(x))^2$ will come into play, and as a result the uncertainty is relatively large.

It is also worthwhile to observe that the expectation $E(x)$ is that choice for which the uncertainty $\int_{\mathbb{R}}(x - E(x))^2|f(x)|^2\,dx$ is the smallest. Indeed, if we try to minimize this quantity by equating to 0 its derivative with respect to $E(x)$, we find that $-2\int_{\mathbb{R}}(x - E(x))|f(x)|^2\,dx = 0$, which gives (2.10). So far, we have discussed the "expectation" and "uncertainty" related to the position of the particle. Of equal relevance are the corresponding notions regarding its momentum. The corresponding rule of quantum mechanics is:

The probability that the momentum y of the particle belongs to the interval (a, b) is $\int_a^b |\hat{f}(y)|^2\,dy$, where $\hat{f}$ is the Fourier transform of f.

Combining these two laws with Theorem 2.17 gives $1/16\pi^2$ as the lower bound for the product of the uncertainty of the position and the uncertainty of the momentum of a particle. So the more certain we are about the location of the particle, the less certain we can be about its momentum, and vice versa. However, we have simplified the statement of the two laws by rescaling to change the units of measurement. Actually, there enters a fundamental but small physical number $\hbar$ called Planck's constant. When properly taken into account, the physical conclusion is

uncertainty principle: (uncertainty of position)$\times$ (uncertainty of momentum) $\geqslant \hbar/16\pi^2$.

2.8. **Central limit theorem.** See page 114–116 in [3]

2.9. **Fast Fourier transform.** See page 224–226 in [12]

3. WAVELET ANALYSIS

Filter theory and signals, applications, TBA

4. APPENDIX 1: DEFINITION OF e, π AND EULER'S FORMULA

4.1. **Definition of e.** Recall that: Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be a linear map (here linear map means $A(au + bv) = aA(u) + bA(v)$ for all a, b in $\mathbb{C}$ and all u, v in $\mathbb{C}^n$). We call $u \neq 0$ in $\mathbb{C}^n$ an *eigenvector* of A if

$$Au = \lambda u, \tag{4.1}$$

where λ is a constant in $\mathbb{C}$.

What is an eigenvector of the derivative ?

By (4.1), fix a complex number λ, we want to find a function $u : \mathbb{R} \to \mathbb{C}$ such that

$$u' = \lambda u.$$

Power series method: Assume that

$$u(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots .$$

Formally, we have

$$u'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} + (n+1) a_{n+1} x^n + \cdots$$

and

$$u' = \lambda u \Leftrightarrow \lambda a_n = (n+1) a_{n+1}, \quad n = 0, 1, \cdots .$$

Thus

$$a_{n+1} = \frac{\lambda a_n}{(n+1)} = \frac{\lambda^2 a_{n-1}}{(n+1)n} = \cdots = \frac{\lambda^{n+1} a_0}{(n+1)n \cdots 1} = \frac{\lambda^{n+1} a_0}{(n+1)!},$$

where we define

$$n! = 1 \cdot 2 \cdots n.$$

Then we have

$$u(x) = u_0 \cdot (1 + \lambda x + \cdots + \frac{(\lambda x)^n}{n!} + \cdots).$$

Put

$$E(x) := 1 + x + \cdots + \frac{x^n}{n!} + \cdots .$$

Since for every $C > 0$,

$$\lim_{n \to \infty} \frac{C^n}{n!} = 0,$$

we know that $E(x)$ converges for all x in $\mathbb{C}$.

Theorem 4.1. *$E(\lambda x)$ is a unique solution of the eigenvalue equation*

$$u' = \lambda u,$$

with initial condition $u(0) = 1$.

Definition 4.1. *We shall define*

$$e := E(1) = 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{n!} + \cdots .$$

4.2. **Definition of the exponential function.** Let us write

$$e^2 = e \cdot e,\ e^3 = e^2 \cdot e,$$

and define e^m inductively by

$$e^{n+1} = e^n \cdot e.$$

Since e is positive, we can take the q-th root of e^m, we write it as $e^{\frac{m}{q}}$. Thus for every $x \in \mathbb{Q}$, e^x is well defined. The following lemma tells us that $E(x)$ is an extension of e^x from $\mathbb{Q}$ to $\mathbb{C}$.

Lemma 4.1. *For every $x \in \mathbb{Q}$, we have $e^x = E(x)$.*

Proof. Since $E(1) = e$, it suffices to prove

$$E(\lambda_1)E(\lambda_2) = E(\lambda_1 + \lambda_2), \tag{4.2}$$

for every λ_1, λ_2 in $\mathbb{C}$. Notice that

$$(E(\lambda_1 x)E(\lambda_2 x))' = E(\lambda_1 x)'E(\lambda_2 x) + E(\lambda_2 x)'E(\lambda_1 x).$$

Put

$$G(x) = E(\lambda_1 x)E(\lambda_2 x).$$

Apply $E(\lambda x)' = \lambda E(\lambda x)$, we get

$$G' = (\lambda_1 + \lambda_2)G.$$

Notice that $G(0) = 1$. Thus Theorem 4.1 implies that

$$G(x) = E((\lambda_1 + \lambda_2)x).$$

Take $x = 1$, we get $E(\lambda_1)E(\lambda_2) = E(\lambda_1 + \lambda_2)$. ☐

Exercise: Find a direct proof of $E(\lambda_1)E(\lambda_2) = E(\lambda_1 + \lambda_2)$ without using Theorem 4.1.

Definition 4.2. *We shall use the same symbol e^x to denote $E(x)$ for all x in $\mathbb{C}$ and call e^x the **exponential function**.*

Remark 1: By Theorem 4.1, we know that e^x is fully determined by

$$(e^x)' = e^x,\ e^0 = 1.$$

Remark 2: Notice that $e > 0$, we know that $e^x > 0$ for all $x \in \mathbb{Q}$, thus $E(x) > 0$ for all $x \in \mathbb{R}$ (since $E(x)$ is smooth and $\mathbb{Q}$ is dense in $\mathbb{R}$). Now we have $E'(x) = E(x) > 0$ for real x, which implies that $E(x)$ is strictly increasing on $\mathbb{R}$. Moreover, we have

$$\lim_{x \to -\infty} E(x) = 0,\ \lim_{x \to \infty} E(x) = \infty,$$

which implies that E maps $\mathbb{R}$ onto $(0, \infty)$. Thus every $x > 0$ has a unique preimage, say $\ln x$, such that $E(\ln x) = x$.

Definition 4.3. *We call $\ln x$, $x > 0$ the **natural logarithmic function**.*

Remark: We have $e^{\ln x} = x$ for every $x \in \mathbb{R}$, moreover $(e^x)' = e^x$ gives

$$(\ln x)' = \frac{1}{x}, \qquad x > 0.$$

4.3. **Definition of π and trigonometric functions.** : Fix $P_0 = (1, 0)$ in the unit circle

$$S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

A counterclockwise rotation of P_0 gives a arc P_0P. The length, say $\theta(P)$, of the arc P_0P is a function of P. It is clear that the circumference diameter ratio is equal to $\theta(-1, 0)$.

Definition 4.4 (Definition of π)**.** *We shall write the circumference diameter ratio as π.*

Denote by

$$F : \theta(P) \mapsto P,$$

the inverse function of $0 \leqslant \theta(P) \leqslant 2\pi$.

Definition 4.5. *We shall write $F(\theta) = (\cos\theta, \sin\theta)$ and call $\cos\theta, \sin\theta$ the* ***trigonometric functions****.*

Notice that

$$F(0) = (1, 0) = F(2\pi), \ F(\pi) = (-1, 0), \ |F(\theta)| \equiv 1.$$

In particular, it gives

$$\sin(0) = \sin(2\pi) = 0, \ \cos(0) = \cos(2\pi) = 1.$$

By definition of θ, we have (since θ is the arclength parameter of S^1)

$$\int_0^{\hat{\theta}} |F'(\theta)| \, d\theta = \hat{\theta}, \ 0 \leqslant \hat{\theta} \leqslant 2\pi,$$

which gives

$$|F'(\theta)| \equiv 1.$$

Now $F(\theta) \cdot F(\theta) \equiv 1$ implies

$$F' \cdot F + F \cdot F' = 2F \cdot F' \equiv 0.$$

Hence $F' \perp F$, thus we know that

$$F'(\theta) = (-\sin\theta, \cos\theta), \ \text{or } F'(\theta) = (\sin\theta, -\cos\theta).$$

But notice that $F'(0) = (0, 1)$, thus we must have

$$F'(\theta) = (-\sin\theta, \cos\theta),$$

which is equivalent to (here we use $i^2 = -1$)

$$(\cos\theta + i\sin\theta)' = i(\cos\theta + i\sin\theta).$$

Notice that $\cos 0 + i \sin 0 = 1$, thus Theorem 4.1 gives

Theorem 4.2 (Euler's formula)**.** $e^{i\theta} = \cos\theta + i\sin\theta$.

Take $\theta = \pi$, we get the following Euler's identity

$$e^{i\pi} = -1.$$

Moreover, apply (4.2), we get

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)},$$

thus by Euler's formula, we have

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2),$$

i.e.

$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2, \tag{4.3}$$

and

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2. \tag{4.4}$$

5. Appendix 2: Lebesgue integral

We shall follow section 1.1 in [3], the readers are recommened to read [13] for the motivations.

5.1. Lebesgue integral on $[-\pi, \pi]$.

Definition 5.1 (Lebesgue measure). *The class of Lebesgue measurable sets in $[-\pi, \pi]$ is the smallest collection, say $\mathcal{L}$, of subsets of $[-\pi, \pi]$ such that*

1) *$[a, b] \in \mathcal{L}$ for all $-\pi \leqslant a < b \leqslant \pi$;*
2) *$E \in \mathcal{L}$ if $|E| := \inf \sum_{n=1}^{\infty} \mathrm{length}(I_n) = 0$, where the infimum is taken over the class of countable coverings of E by means of open intervals I_n (i e $\cup_{n=1}^{\infty} I_n \supset E$);*
3) *$\mathcal{L}$ is closed under countable unions, countable intersections and complementation.*

We call $|E|$ the Lebesgue measure of E if E is Lebesgue measurable.

Remark: Without the second assumption, the above definition gives Borel measurable sets.

Definition 5.2 (measurable function). *We call a real function $f : [-\pi, \pi] \to \mathbb{R}$ a measurable function if $f^{-1}((a, b])$ are Lebesgue measurable for all real numbers $a < b$.*

Remark: A complex function is said to be Lebesgue measurable if both its real and imaginary parts are Lebesgue measurable. We shall denote by $\mathcal{M}[-\pi, \pi]$ the space of all complex Lebesgue measurable functions.

Definition 5.3 (Lebesgue integral of a nonnegative measurable function). *Let $f : [-\pi, \pi] \to [0, \infty)$ be a measurable function. We call*

$$\int_{-\pi}^{\pi} f(x)\,dx := \lim_{n\to\infty} \sum_{k=0}^{\infty} k2^{-n} \left| f^{-1}(A_k) \right|, \qquad A_k := (k2^{-n}, (k+1)2^{-n}],$$

the Lebesgue integral of f. We say that f is integrable if $\int_{-\pi}^{\pi} f(x)\,dx < \infty$.

Remark: A general real measurable function f is integrable if both $\max\{f, 0\}$ and $\max\{-f, 0\}$ are integrable. A complex measurable function is integrable if both its real and imaginary parts are integrable.

Now let us define the L^p space $p \geqslant 1$. First we say that $f = g$ *almost everywhere* in $[-\pi, \pi]$ if $|\{f \neq g\}| = 0$.

Definition 5.4. *Fix $p \geqslant 1$, we shall define*

$$L^p[-\pi,\pi] := \{f \in \mathcal{M}[-\pi,\pi] : \int_{-\pi}^{\pi} |f|^p \, dx < \infty\}/\sim,$$

where

$$f \sim g \Leftrightarrow f = g \ \text{ a.e. on } [-\pi,\pi].$$

5.2. **Lebesgue measure on $\mathbb{R}^n$.** Just replace interval by n times product of intervals. We only give the definition of Lebesgue measurable sets in $\mathbb{R}^n$ and leave the other definitions to the readers.

Definition 5.5. *The class of Lebesgue measurable sets in $\mathbb{R}^n$ is the smallest collection, say $\mathcal{L}$, of subsets of $\mathbb{R}^n$ such that*

1) *$\mathcal{L}$ contains all closed n-cubes;*
2) *$E \in \mathcal{L}$ if $|E| := \inf \sum_{n=1}^{\infty} \text{length}(I_n) = 0$, where the infimum is taken over the class of countable coverings of E by means of open n-cubes I_n (i e $\cup_{n=1}^{\infty} I_n \supset E$);*
3) *$\mathcal{L}$ is closed under countable unions, countable intersections and complementation.*

We call $|E|$ the Lebesgue measure of E if E is Lebesgue measurable.

Example: Lebesgue integral of $e^{-\phi}$:

$$I(\phi) := \int_{\mathbb{R}^n} e^{-\phi(x)} \, dx_1 \cdots dx_n,$$

where $\phi(x)$ is a real Lebesgue measurable function.

Theorem 5.1. *Assume that for every real number s, $|\Omega(s)| < \infty$, where*

$$\Omega(s) := \{x \in \mathbb{R}^n : \phi(x) < s\}.$$

Then we have

$$I(\phi) = \int_{-\infty}^{\infty} |\Omega(s)| \, e^{-s} \, ds.$$

Proof. By definition of the Lebesgue integral, we have

$$I(\phi) = \lim_{n \to \infty} \sum_{k=0}^{\infty} k2^{-n} |\{k2^{-n} < e^{-\phi} \leqslant (k+1)2^{-n}\}|.$$

Since each $|\Omega(s)|$ is finite, we can write $|\{k2^n < e^{-\phi} \leqslant (k+1)2^n\}|$ as

$$|\Omega(-\log(k2^{-n}))| - |\Omega(-\log((k+1)2^{-n}))|.$$

Thus the above limit of sums can be written as (try!)

$$I(\phi) = \lim_{n \to \infty} \sum_{k=1}^{\infty} 2^{-n} |\Omega(-\log(k2^{-n}))|.$$

Notice that

$$t \mapsto |\Omega(-\log t)|, \ \ t > 0,$$

is a positive locally bounded decreasing function on $0 < t < \infty$, thus it is Riemann integrable (see Proposition 1.3 in [12]). We know that the above limit is equal to the following Riemann integral

$$\int_0^\infty |\Omega(-\log t)| \, dt.$$

Thus our formula follows by a change of variable $t = e^{-s}$. □

Remark: We can also get a similar formula for a general Lebesgue measurable set in $\mathbb{R}^n$ (not only on $\mathbb{R}^n$). For example, if a function is well defined and negative on a Lebesgue measurable set Ω, extend ϕ such that $\phi = \infty$ outside Ω then

$$\Omega(s) = \Omega, \qquad \forall \, s \geqslant 0.$$

Thus the above theorem gives

$$I(\phi) = \int_{-\infty}^0 |\Omega(s)| \, e^{-s} \, ds + |\Omega| \cdot \int_{-\infty}^0 e^{-s} \, ds.$$

Notice that $\int_{-\infty}^0 e^{-s} \, ds = 1$, hence we have

$$\int_\Omega e^{-\phi(x)} \, dx_1 \cdots dx_n = \int_{-\infty}^0 |\Omega(s)| \, e^{-s} \, ds + |\Omega|,$$

in case $\phi < 0$ in Ω.

6. Exercise sets

6.1. **Exercise set 1: Fejér kernel and its applications.** — From page 53–58, page 63 in [12], see also page 34–36 in [3].

6.1.1. *Fejér's theorem.* The aim is to prove the following theorem:

Theorem 6.1 (Fejér's theorem). *If $f \in PC^0(S^1)$ then*

$$\lim_{N \to \infty} \left| \frac{f_0(x) + \cdots + f_{N-1}(x)}{N} - \frac{f(x+) + f(x-)}{2} \right| = 0,$$

By definition we have (see (1.2))

$$f_N(x_0) = (f, D_N(x - x_0)),$$

where the Dirichlet kernel $D_N(x)$ is defined by

$$D_N(x) := \sum_{n=-N}^{N} \omega^n, \qquad \omega := e^{ix}.$$

Thus we have

$$\frac{f_0(x_0) + \cdots + f_{N-1}(x_0)}{N} = (f, F_N(x - x_0)),$$

where

$$F_N(x) := \frac{\sum_{n=0}^{N-1} D_n(x)}{N},$$

is called the *Fejér kernel*. We have proved that (see the proof of (1.1))

$$D_n(x) = \frac{\omega^{-n} - \omega^{n+1}}{1-\omega},$$

Exercise 1: Prove the following facts on the Fejér kernel

1) $F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}$;
2) $\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)\, dx = 1$.

Put

$$I_N(x) := \frac{f_0(x) + \cdots + f_{N-1}(x)}{N} - \frac{f(x+) + f(x-)}{2}.$$

Use *Exercise 1* to prove that:

Exercise 2: Assume further that f is continuous at x_0. Then

1) $I_N(x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x) - f(x_0))\, \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}\, dx$;
2) For every $0 < \delta < \pi$, we have $\lim_{N\to\infty} \int_{\delta \leqslant |x| \leqslant \pi} \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}\, dx = 0$;
3) $\lim_{N\to\infty} I_N(x_0) = 0$.

Exercise 3: Use *Exercise 2* and the method in section 1.4.4 to prove Fejér's theorem.

6.1.2. *Weierstrass approximation theorem.* In case $f \in C^0(S^1)$, Fejér's theorem implies that the limit of the following trigonometric polynomials

$$\frac{f_0(x) + \cdots + f_{N-1}(x)}{N}$$

is f. Thus continuous functions on the circle can be uniformly approximated by trigonometric polynomials. Using the fact that e^{ix} *can be approximated by polynomials uniformly on any intervel* to show the following *Weierstrass approximation theorem*:

Exercise 4: Let $a < b$ be two real numbers and f be a continous function on $[a, b]$. Then for every $\varepsilon > 0$ there exists a polynomial P such that

$$\sup_{x\in[a,b]} |f(x) - P(x)| < \varepsilon.$$

6.1.3. *Poisson kernel.* — From page 36 in [3], see also page 54–58 in [12].

Recall the definition of the *Poisson kernel* $P_r(\theta)$ (see Example 4 in section 1.4) defined for $\theta \in [-\pi, \pi]$ and $0 \leqslant r < 1$ by the following uniformly convergent series

$$P_r(\theta) = \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta}.$$

Check that:

Exercise 5: We can write the Poisson kernel as

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2},$$

and for every $f \in C^0(S^1)$,

$$\sum_{n\in\mathbb{Z}} \hat{f}(n) r^{|n|} e^{inx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) P_r(x-\theta)\, d\theta.$$

Using the above exercise and the proof of Fejér's theorem to prove:

Exercise 6: For every $f \in PC^0(S^1)$, we have

$$\lim_{r\to 1} \sum_{n\in\mathbb{Z}} \hat{f}(n) r^{|n|} e^{inx} = \frac{f(x+) + f(x-)}{2}.$$

6.2. **Exercise set 2: Gibbs' Phenomenon.** — From page 94 in [12], see also page 44–46 in [3].

Let $f(x)$ denotes the sawtooth function defined by

$$f(x) = \frac{\pi - x}{2}$$

on the interval $(0, 2\pi)$ with $f(0) = 0$ and extended by periodicity to all of $\mathbb{R}$.

Exercise 1: Show that $f(0+) = \frac{\pi}{2}$, $f(0-) = -\frac{\pi}{2}$ and the Fourier series of f *on* $(0, 2\pi)$ is

$$f(x) \sim \frac{1}{2i}\sum_{|n|\neq 0} \frac{e^{inx}}{n} = \sum_{n=1}^{\infty} \frac{\sin nx}{n}.$$

Exercise 2: By the above exercise, we have $f_N(x) = \frac{1}{2i}\sum_{0<|n|\neq N} \frac{e^{inx}}{n}$. Show that

$$f_N(x) = \sum_{n=1}^{N} \frac{\sin nx}{n} = \frac{1}{2}\int_0^x (D_N(t) - 1)\ dt.$$

Hint: first prove that $D_N(t) - 1 = 2(\cos t + \cdots + \cos Nt)$. and for $N \geqslant 2$, $f_N(x)$ is increasing on $[0, \pi/N]$. (Hint: use the integral formula to compute the derivative).

Exercise 3: Use *Exercsie 2* to show that, for $0 < x < 2\pi$, we have

$$f_N(x) - f(x) = \frac{1}{2}\int_0^x D_N(t)\, dt - \frac{\pi}{2}.$$

Recall that

$$D_N(t) = \frac{\sin(N + \frac{1}{2})t}{\sin\frac{t}{2}},$$

show that

$$\left|D_N(t) - \frac{\sin(N+\frac{1}{2})t}{t/2}\right| \leqslant \left|\frac{t - 2\sin\frac{t}{2}}{t\sin\frac{t}{2}}\right| \to 0,$$

when $t \to 0+$.

Remark: The above exercise says that when x is small enough the error term $f_N(x) - f(x)$ is well approximated (uniformly in N) by

$$\frac{1}{2}\int_0^x \frac{\sin(N+\frac{1}{2})t}{t/2}\,dt - \frac{\pi}{2},$$

Exercise 4: Put

$$y(x) = \frac{1}{2}\int_0^x \frac{\sin(N+\frac{1}{2})t}{t/2}\,dt.$$

Show that $y'(x) = 0, x > 0$ if and only if $(N+\frac{1}{2})x = k\pi$ for some positive integer k.

Remark: Since $\frac{1}{t}$ is decreasing on $(0,\infty)$ and $\sin(N+\frac{1}{2})t$ is periodic, after a few reflections (try), you might get that $y(x)$ achieves its maximum of at the first zero point of y', i e

$$\sup_{0<x<\infty} y(x) = y\left(\frac{\pi}{N+\frac{1}{2}}\right)$$

Exercise 5: Show that

$$\frac{1}{2}\int_0^{\frac{\pi}{N+\frac{1}{2}}} \frac{\sin(N+\frac{1}{2})t}{t/2}\,dt = \int_0^\pi \frac{\sin x}{x}\,dx.$$

Remark: The above exercises show that the "overshoot" of f at the origin defined by

$$\lim_{\varepsilon\to 0+}\lim_{N\to\infty}\sup_{0<x<\varepsilon}(f_N(x)-f(x)) = \int_0^\pi \frac{\sin x}{x}\,dx - \frac{\pi}{2},$$

is positive. In fact, we have

$$\int_0^\pi \frac{\sin x}{x}\,dx - \frac{\pi}{2} \sim (0.089490)\frac{\pi}{2}.$$

Since the "jump" of f at the origin defined by

$$\frac{f(0+)-f(0-)}{2}$$

is equal to $\frac{\pi}{2}$, we proved the following

Gibbs' Phenomenon: *Near a jump discontinuity, the Fourier series of a function overshoots (or undershoots) it by approximately* 9% *of the jump.*

6.3. **Exercise set 3: Isoperimetric inequality, Eigenfunction expansion and temperature of the earth.**

6.3.1. *Isoperimetric inequality.* See Exercise 1,2,3, page 120–122 in [12].

6.3.2. *Eigenfunction expansion for the Laplacian.* See page 56–60 in [3]

6.3.3. *Temperature of the earth.* See page 68–70 in [3]

6.4. **Exercise set 4: Eigenfunctions of the Fourier transform.** See page 97–101 in [3]. A canonical basis of $L^2(\mathbb{R})$.

6.5. **Exercise set 5: Poisson summation formula.** see page 153–158 in [12].

7. TMA4170 (2019 SPRING) EXERCISE

7.1. **Week 2.** The main aim is to review the definition of e, π and the application of the Euler formula in the Appendix 1.

Exercise 1: Find the definition of e^x, $\cos x, \sin x$ and the Euler formula in the Appendix 1, use it to prove the followings:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \ \ \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

and

$$\lim_{x \to 0} \frac{e^{ix} - 1}{x} = i.$$

Exercise 2: Read Appendix 1 and find the proof of the following formula

$$e^{\lambda_1 + \lambda_2} = e^{\lambda_1} e^{\lambda_2},$$

for every complex numbers $\lambda_1 + \lambda_2$. Use it to show

1) $e^{x+iy} = e^x(\cos y + i \sin y)$ and $|e^{x+iy}| = e^x$ for all real numbers x, y;
2) for a complex number z, $e^z = 1$ if and only if $z = 2\pi k i$ for some integer k;
3) for a complex number z, $\sin z = 0$ if and only if $z = \pi k$ for some integer k.

Exercise 3: (From week 2 exercsie in 2014). Compute the Fourier series expansion of

$$f(x) = (\sin x)^3,$$

on $[0, 2\pi]$. Can you do it without computing the integral ?

Exercise 4: With the inner product $(f, g) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f \bar{g} \, dx$, show that

$$(\cos nx, \cos nx) = (\sin nx, \sin nx) = \frac{1}{2}, \ \ (\cos nx, \sin mx) = 0,$$

for all positive integers n, m; assume further that $n \neq m$, show that

$$(\cos nx, \cos mx) = (\sin nx, \sin mx) = 0.$$

Exercise 5: Write $z = re^{i\theta}$, show that the Poisson kernel

$$P_r(\theta) = \sum_{n \in \mathbb{Z}} r^{|n|} e^{in\theta}, \ \ 0 \leqslant r < 1, \ \theta \in [-\pi, \pi]$$

can be written as

$$P_r(\theta) = \frac{1 - |z|^2}{|1 - z|^2} = \mathrm{Re} \, \frac{1 + z}{1 - z}.$$

7.2. **Week 3.** For section 1.1–1.5, taken from Exercise 1, 2, 4, 8, 10 in [12] (page 58–61).

Exercise 1: Let $f \in PC^0(S^1)$, show that

$$\int_a^b f(x)\,dx = \int_{a+2\pi}^{b+2\pi} f(x)\,dx$$

and

$$\int_{-\pi}^{\pi} f(x+a)\,dx = \int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi+a}^{\pi+a} f(x)\,dx.$$

Exercise 2: Let $f \in PC^0(S^1)$. Show that

1) We can write $f_N(x) = \sum_{|n|\leqslant N} \hat{f}(n)e^{inx}$ as
$$f_N(x) = \hat{f}(0) + \sum_{n=1}^{N} \Big((\hat{f}(n) + \hat{f}(-n)) \cos nx + i(\hat{f}(n) - \hat{f}(-n)) \sin nx \Big);$$
2) If f is even then $\hat{f}(n) = \hat{f}(-n)$ and we get a cosine series;
3) If f is odd then $\hat{f}(n) = -\hat{f}(-n)$ and we get a sine series;
4) If f is π-periodic then $\hat{f}(n) = 0$ for all odd n;
5) Assume further that $f \in C^1(S^1)$. Show that f is real valued if and only if $\overline{\hat{f}(n)} = \hat{f}(-n)$ for all n.

Exercise 3: Consider the 2π-periodic odd function defined on $[0, \pi]$ by $f(x) = x(\pi - x)$. Draw the graph of f, compute the Fourier series of f then show that

$$f(x) = \frac{8}{\pi} \sum_{k \text{ odd } \geqslant 1} \frac{\sin kx}{k^3}.$$

Exercise 4: Verify that $\frac{1}{2i} \sum_{n \neq 0} \frac{e^{inx}}{n}$ is the Fourier series of the 2π-periodic *sawtooth* function defined by $f(0) = 0$ and

$$f(x) = -\frac{\pi}{2} - \frac{x}{2}, \quad -\pi \leqslant x < 0; \quad f(x) = \frac{\pi}{2} - \frac{x}{2}, \quad 0 < x \leqslant \pi.$$

Show that the series $\{a_N(x)\}$ defined by

$$a_N(x) := \sum_{n=1}^{N} \frac{\sin nx}{n},$$

converges for all $x \in \mathbb{R}$.

Exercise 5: Let $f \in PC^1(S^1)$. Show that there exists a positive constant C such that $|\hat{f}(n)| \leqslant C|n|^{-1}$. Assume futher that $f \in C^k(S^1)$ $(k \geqslant 2)$, show that $\hat{f}(n) \leqslant C|n|^{-k}$ for a positive constant C. Find $f \in PC^\infty(S^1)$ such that $\{\hat{f}(n)n^2\}_{n\in\mathbb{Z}}$ is unbounded.

Remark: One may also prove that if f is continuous and $f \in PC^2(S^1)$ then $\{\hat{f}(n)n^2\}_{n\in\mathbb{Z}}$ is bounded (try!). So the example that you find must be non-continuous.

7.3. **Week 4.** For "Lebesgue integral", "Gaussian integral" and orthogonal projections.

Exercise 1: Use Theorem 5.1 to compute

$$\int_{\mathbb{R}^2} e^{-x^2-y^2}\, dxdy.$$

Use the vaule of the above integral and the Fubini theorem to prove the following famous "Gaussian" integral identity

$$\int_{\mathbb{R}} e^{-x^2}\, dx = \sqrt{\pi}.$$

Exercise 2: Use Theorem 5.1 to show that if ϕ is a real Lebesgue measurable function on a Lebesgue measurable set Ω in $\mathbb{R}^n$ such that $\phi < c$ in Ω then

$$\int_{\Omega} e^{-\phi}\, dx_1 \cdots dx_n = \int_{-\infty}^{c} |\Omega(s)| e^{-s}\, ds + e^{-c} \cdot |\Omega|.$$

Exercise 3: Show that the Lebesgue measure of every countable set in $\mathbb{R}$ is zero. Let E be the set of irrational numbers in $[0,1]$. Show that $|E| = 1$.

Exercise 4, 5: See week 5 exercise in 2016, problem 1, 2.

1. Describe how to find the minimum

$$\min \int_{-1}^{1} (\sin x - P(x))^2\, dx$$

taken among all polynomials P of degree no bigger than 3.

$\star\star$ 2. Let $\mathrm{sgn}(x)$ be the sign function defined by

$$\mathrm{sgn}(x) = -1,\ x<0;\ \ \mathrm{sgn}(x) = 1,\ x>0;\ \ \mathrm{sgn}(x) = 0,\ x=0.$$

The Rademacher functions are defined by

$$\phi_n := \mathrm{sgn}(\sin(2^n \pi x)),\ \ n = 0, 1 \cdots .$$

Show that $\{\phi_n\}_{n\geqslant 0}$ is an orthonormal system in $L^2[0,1]$ but is NOT an orthonormal basis of $L^2[0,1]$.

7.4. **Week 5-1.** The aim is to prove the following theorem:

Theorem 7.1 (Fejér's theorem)**.** *If $f \in PC^0(S^1)$ then*

$$\lim_{N\to\infty} \left| \frac{f_0(x) + \cdots + f_{N-1}(x)}{N} - \frac{f(x+) + f(x-)}{2} \right| = 0,$$

By definition we have (see (1.2))

$$f_N(x_0) = (f, D_N(x - x_0)),$$

where the Dirichlet kernel $D_N(x)$ is defined by

$$D_N(x) := \sum_{n=-N}^{N} \omega^n, \qquad \omega := e^{ix}.$$

Thus we have

$$\frac{f_0(x_0) + \cdots + f_{N-1}(x_0)}{N} = (f, F_N(x - x_0)),$$

where

$$F_N(x) := \frac{\sum_{n=0}^{N-1} D_n(x)}{N},$$

is called the *Fejér kernel*. We have proved that (see the proof of (1.1))

$$D_n(x) = \frac{\omega^{-n} - \omega^{n+1}}{1 - \omega},$$

Exercise 1: Prove the following facts on the Fejér kernel

1) $F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}$;
2) $\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)\,dx = 1$.

Put

$$I_N(x) := \frac{f_0(x) + \cdots + f_{N-1}(x)}{N} - \frac{f(x+) + f(x-)}{2}.$$

Use *Exercise 1* to prove that:

Exercise 2: Assume further that f is continuous at x_0. Then

1) $I_N(x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x) - f(x_0))\,\frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}\,dx$;
2) For every $0 < \delta < \pi$, we have $\lim_{N\to\infty}\int_{\delta\leqslant|x|\leqslant\pi} \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}\,dx = 0$;
3) $\lim_{N\to\infty} I_N(x_0) = 0$.

Exercise 3: Use *Exercise 2* and the method in section 1.4.4 to prove Fejér's theorem.

Weierstrass approximation theorem: In case $f \in C^0(S^1)$, Fejér's theorem implies that the limit of the following trigonometric polynomials

$$\frac{f_0(x) + \cdots + f_{N-1}(x)}{N}$$

is f. Thus continuous functions on the circle can be uniformly approximated by trigonometric polynomials. Using the fact that e^{ix} *can be approximated by polynomials uniformly on any interval* to show the following *Weierstrass approximation theorem*:

Exercise 4: Let $a < b$ be two real numbers and f be a continous function on $[a, b]$. Then for every $\varepsilon > 0$ there exists a polynomial P such that

$$\sup_{x\in[a,b]} |f(x) - P(x)| < \varepsilon.$$

Poisson kernel: Recall the definition of the *Poisson kernel* $P_r(\theta)$ (see Example 4 in section 1.4) defined for $\theta \in [-\pi, \pi]$ and $0 \leqslant r < 1$ by the following uniformly convergent series

$$P_r(\theta) = \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta}.$$

We have the following close formula for the Poisson kernel as

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2},$$

which implies the *positivity* of the Poisson kernel.

Exercise 5: For every $f \in C^0(S^1)$, prove that

$$\sum_{n\in\mathbb{Z}} \hat{f}(n) r^{|n|} e^{inx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) P_r(x-\theta)\, d\theta,$$

then use the proof of Fejér's theorem to show, for every $f \in PC^0(S^1)$, we have

$$\lim_{r\to 1} \sum_{n\in\mathbb{Z}} \hat{f}(n) r^{|n|} e^{inx} = \frac{f(x+) + f(x-)}{2}.$$

7.5. **Week 5-2.** The Fourier transform on $\mathbb{R}^n$ is defined by

$$\hat{f}(\gamma) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\gamma}\, dx, \ \ dx := dx_1 \cdots dx_n,$$

where

$$x\cdot\gamma := x_1\gamma_1 + \cdots + x_n\gamma_n.$$

They we can similarly define the Schwartz space $\mathcal{S}$ and prove the Fourier inversion formula and the Plancherel identity, moreover, the convolution is also well defined for functions in $\mathcal{S}$.

Exercise 1: Fix $t > 0$ and put $f(x) = e^{-t\pi|x|^2}$, show that

$$\hat{f}(\gamma) := t^{-\frac{n}{2}} e^{-\frac{\pi|\gamma|^2}{t}}.$$

We call the above function the n-dimensional heat kernel on $\mathbb{R}^n$

Remark: The Poisson summation formula together with the above exercise give that

$$\sum_{n\in\mathbb{Z}^n} e^{-t\pi|x+n|^2} = \sum_{n\in\mathbb{Z}^n} t^{-\frac{n}{2}} e^{-\frac{\pi|\gamma+n|^2}{t}},$$

The left hand side is called *heat kernel on the n-dimensional torus* (see Definition 1.7). Poisson summation formula gives a nice representation of the torus heat kernel by the $\mathbb{R}^n$ heat kernel.

The following three exercises come from [12], page 161–165, Exercise 1,2, 15.

Exercise 2: Suppose f is a smooth function on $\mathbb{R}$ that vanishes outside the interval $[-M, M]$. 1) Fix L with $L/2 > M$, and show that $f(x) = \sum a_n(L) e^{2\pi i n x/L}$ where

$$a_n(L) = \frac{1}{L}\int_{-L/2}^{L/2} f(x) e^{-2\pi i n x/L}\, dx = \frac{1}{L}\hat{f}(n/L).$$

2) Show that for an arbitrary continuous function F on $\mathbb{R}$ such that $|F(y)| \leqslant C(1+y^2)^{-1}$, we have

$$\int_{\mathbb{R}} F(y)\, dy = \lim_{\delta\to 0,\ \delta>0} \delta \sum_{n=-\infty}^{\infty} F(\delta n).$$

3) Conclude that $f(x)=\int_{\mathbb{R}}\hat{f}(y)e^{2\pi i x y}\,dy$.

Exercise 3: Let f and g be the functions defined by

$$f(x)=1,\ \text{ if } |x|<1; \quad f(x)=0,\ \text{ otherwise}$$

and

$$g(x)=1-|x| \ \text{ if } |x|<1; \quad g(x)=0,\ \text{ otherwise},$$

Although f is not continuous, the integral defining its Fourier transform still makes sense. Show that

$$\hat{f}(\gamma)=\frac{\sin 2\pi\gamma}{\pi\gamma},\ \hat{g}(\gamma)=\left(\frac{\sin \pi\gamma}{\pi\gamma}\right)^2,$$

and with the understanding that $\hat{f}(0)=2$ and $\hat{g}(0)=1$.

Exercise 4: Apply the Poisson summation formula to function g in the above exercise to obtain

$$\sum_{n\in\mathbb{Z}}\frac{1}{(n+\alpha)^2}=\frac{\pi^2}{(\sin\pi\alpha)^2}$$

whenever α is real but not equal to an integer. Then prove as a consequence that

$$\sum_{n\in\mathbb{Z}}\frac{1}{(n+\alpha)}=\frac{\pi}{\tan\pi\alpha}.$$

whenever α is real but not equal to an integer. *Hint: first prove it when $0<\alpha<1$. To do so, integrate the first formula. What is the precise meaning of the series on the left-hand side of the second formula? Evaluate at $\alpha=1/2$.*

The final exercise is on *elementary solution for constant coefficient PDEs*.

****Exercise 5**: Fix $s>0$ and $\gamma\in\mathbb{R}$. Compute the following integral (it is called Laplace transform of the heat kernel)

$$G(\gamma;s):=\int_0^\infty e^{-st}t^{-\frac{1}{2}}e^{-\frac{\pi\gamma^2}{t}}\,dt,$$

and check that

$$\int_{\gamma\in\mathbb{R}}G(\gamma;s)\left(sf(\gamma)-\frac{1}{4\pi}f''(\gamma)\right)d\gamma=f(0) \tag{7.1}$$

for every smooth function f on $\mathbb{R}$ that vanishes outside a bounded interval.

Hint: one may compute the integral directly, but it is not easy (try!). Another approach is to apply Exercise 1 to show

$$G(\gamma;s)=\int_{\mathbb{R}}\frac{1}{s+\pi x^2}e^{-2\pi i x\gamma}\,dx,$$

then use the Cauchy integral formula.

Remark: Using the language of distribution we can write (7.1) as

$$\left(s-\frac{1}{4\pi}\frac{d^2}{d\gamma^2}\right)G(\gamma;s)=\delta_0,$$

In general, consider a partial differential operator

$$P := \sum c_\alpha D^\alpha, \; D^\alpha := \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{(\partial x_1)^{\alpha_1}\cdots(\partial x_n)^{\alpha_n}}$$

where c_α are complex constants. We call a distribution u an *elementary solution for* P if

$$P(u) = \delta_0,$$

in the sense of distribution. Apply the Fourier transform to the above equation we get

$$\widehat{P(u)} = \widehat{\delta_0} = 1.$$

Since

$$\widehat{P(u)} = \sum c_\alpha (2\pi i \gamma)^\alpha \hat{u},$$

we must have

$$\hat{u} = \frac{1}{\tilde{P}(\gamma)}, \quad \tilde{P}(\gamma) := \sum c_\alpha (2\pi i\gamma)^\alpha.$$

Thus the Fourier inversion formula gives the following Fourier type elementary solution

$$u(x) = \int_{\mathbb{R}^n} \frac{1}{\tilde{P}(\gamma)} e^{2\pi i x\cdot\gamma}\, d\gamma.$$

Hörmander [10] *proved that* $\frac{1}{\tilde{P}(\gamma)}$ *is always a well defined tempered distribution, thus every* P *above has a tempered distribution elementary solution.*

7.6. **Week 6.** *Complex analysis results used in the proof of the Paley–Wiener theorem.*

Let us *identify* a smooth *closed* curve in $\mathbb{C}$ as a smooth 2π-periodic function $\gamma(t)$ from $\mathbb{R}$ to $\mathbb{C}$, more precisely, the curve associated to γ is the following set

$$\{\gamma(t) \in \mathbb{C} : t \in [0, 2\pi)\},$$

with orientation from 0 to 2π. The fundamental theorem in complex analysis is the following:

Theorem 7.2 (Cauchy integral theorem)**.** . *Let* f *be a holomorphic function on an open set, say* U*, in* $\mathbb{C}$*. Assume that inside* U *we can move a smooth* closed *curve* γ_0 *smoothly to another smooth* closed *curve* γ_1*. Then*

$$\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz$$

Proof. Divide the proof into several exercises! □

1. Cauchy integral formula, distribution formulation $\partial/\partial\bar{z}(z^{-1}) =?$ and the Stokes formula
2. Mean–value property and maximum principle

7.7. **Week 7.** Exercise set 4.

7.8. **Week 8.** Exercise set 5

7.9. **Week 9.** *Uncertainty principle in terms of the Hermite operator*.

The Heisenberg uncertainty principle can be formulated in terms of the operator

$$Lf := -f'' + x^2 f,$$

which acts on Schwartz space $\mathcal{S}$. This operator, sometimes called the Hermite operator, is the quantum analogue of the *harmonic oscillator*. Consider the usual inner product on $\mathcal{S}$ given by

$$(f, g) := \int_{\mathbb{R}} f\bar{g}\, dx,$$

7.10. **Week 10-11.** Exercise set 1-3.

7.11. **Week 12-15.** Wavelet

REFERENCES

[1] A. Boggess and F. Narcowich, *A first course in Wavelets with Fourier Analysis*, Wiley, 2nd Edition, 2009.
[2] P. R. Chernoff, *Pointwise convergence of Fourier series*, The American Mathematical Monthly, **87** (1980), 399–400.
[3] H. Dym and H. P. McKean, *Fourier Series and Integrals*, Academic Press, 1972.
[4] F. Diamond and J. M. Shurman, *A first course in modular forms*, Vol. 228. New York: Springer, 2005.
[5] W. Feller, *An Introduction to Probability Theory and its Applications*, Volume I, 3rd edition (1968).
[6] G. W. Ford and G. E. Uhlenbeck, *Lectures in statistical mechanics*, 1974
[7] R. Gardner, *The Brunn–Minkowski inequality*, Bulletin of the American Mathematical Society 39.3 (2002): 355–405.
[8] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer.
[9] L. Hörmander, *Linear Partial Differential Operators*, Springer.
[10] L. Hörmander, *On the division of distributions by polynomials*. Arkiv för matematik,(1958) 555–568.
[11] K. Seip, *The Paley–Wiener theorem and Fourier analysis*, see https://wiki.math.ntnu.no/_media/tma4170/2018v/pwfinal.pdf
[12] E. Stein and R. Shakarchi, *Fourier analysis*, Princeton lectures in analysis I, 2003.
[13] E. Stein and R. Shakarchi, *Real analysis*, Princeton lectures in analysis III, 2003.
[14] E. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University press.
[15] T. Tao, *The Schrödinger equation*, see http://www.math.ucla.edu/∼tao/preprints/schrodinger.pdf
[16] T. Tao, *https://terrytao.wordpress.com/2018/01/19/the-de-bruijn-newman-constant-is-non-negativ/*

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