WEEK 9

This exercise set is on eigentheory of the Fourier transform (see page 173 in [3]). For notations, see the second part of TMA4170 lecture notes.

Exercise 1: We have already proved that the Hermite functions h_k are orthogonal to each other and satisfy

$$\sum_{k=0}^{\infty} h_k(x)\frac{t^k}{k!} = e^{-\pi(x+2t)^2} \cdot e^{2\pi t^2}.$$

Use this formula to prove that

$$||h_k||^2 := \int_{\mathbb{R}} |h_k(x)|^2 \, dx = 2^{-1/2}(4\pi)^k k!.$$

Exercise 2: Use integration by parts to show that

$$((A-B)h_{k-1}, (A-B)h_{k-1}) = (-(A+B)(A-B)h_{k-1}, h_{k-1}),$$

i.e. the adjoint operator af $A-B$ is $-(A+B)$.

Exercise 3: Recall that $h_k = (A-B)h_{k-1}$ and

$$(A+B)(A-B) = 4\pi\mathfrak{f} - 2\pi.$$

Use Exercise 2 and

$$\mathfrak{f}(h_{k-1}) = -(k-\frac{1}{2})h_{k-1}$$

to prove that

$$||h_k||^2 = 4\pi k||h_{k-1}||^2.$$

Then use it to evaluate $||h_k||^2$ a second way.

Remark: Notice that $\mathfrak{f}(h_k) = -(k+\frac{1}{2})h_k$ is equivalent to

$$-(h_k)''(x) + 4\pi^2 x^2 h_k(x) = 2\pi(2k+1)h_k(x).$$

Consider

$$e_k(x) = \frac{h_k((2\pi)^{-1/2}x)}{||h_k((2\pi)^{-1/2}x)||} = \frac{h_k((2\pi)^{-1/2}x)}{\pi^{1/4}\sqrt{(4\pi)^k k!}},$$

we know that $\{e_k\}$ is an orthonormal basis of $L^2(\mathbb{R})$. Try to do the following two exercises.

Exercise 4: Put $Lf = -f'' + x^2 f$. Show that

$$L(e_k) = (2k+1)e_k, \quad k = 0, 1, 2, \cdots.$$

Exercise 5: One version of the Heisenberg uncertainty principle (see Week 8 Exercise 2) is:

$$(Lf, f) = ||f'||^2 + ||xf||^2 \geq ||f||^2, \ \forall\, f \in \mathcal{S}.$$

Try to prove the following effective version

$$(Lf, f) = ||f'||^2 + ||xf||^2 = \sum_{n=0}^{\infty}(2n+1)|(f, e_n)|^2 \geq \sum_{n=0}^{\infty}|(f, e_n)|^2 = ||f||^2,$$

for all $f \in \mathcal{S}$. In particular, the above formula gives

$$||f'||^2 + ||xf||^2 \geq ||f||^2 + 2 \cdot \mathrm{Distance}\{f, \mathbb{C}e_0\}^2,$$

where $\mathrm{Distance}\{f, \mathbb{C}e_0\}$ denotes the L^2 distance from f to the line $\mathbb{C}e_0$ generated by the Gaussian function e_0.

Remark: For more information on the above Hermite operator L and its modern descriptions, see [2]. For an effective version of the original Heisenberg uncertainty principle, say page 117 in [1].

REFERENCES

[1] H. Dym and H. P. McKean, *Fourier Series and Integrals*, Academic Press, 1972.
[2] R. Howe, *On the role of the Heisenberg group in harmonic analysis*, Bull. Amer. Math. Soc. (1980).
[3] E. Stein and R. Shakarchi, *Fourier analysis*, Princeton lectures in analysis.