WEEK 8

Sampling theorem for $g \in L^2[a,b]$:

Exercise 1: Assume that $g \in L^2[a,b]$, where $b > a$ are real constants. Try to prove the following orthogonal decomposition type sampling formula

$$\hat{g}(x) = \sum_{n\in\mathbb{Z}} \hat{g}\left(\frac{n}{b-a}\right) \frac{\sin \pi(n-(b-a)x)}{\pi(n-(b-a)x)} e^{\pi i \frac{b+a}{b-a}(n-(b-a)x)}, \quad \forall\, x \in \mathbb{R}$$

and

$$||g||^2 = ||\hat{g}||^2 = \frac{1}{b-a} \sum_{n\in\mathbb{Z}} |\hat{g}\left(\frac{n}{b-a}\right)|^2.$$

Hint: apply the sampling theorem to the following function in $L^2[-1/2, 1/2]$

$$f(x) := g\left(\frac{a+b}{2} + (b-a)x\right).$$

Uncertainty principle in terms of the Hermite operator L, from page 168–169 in [2]: The Heisenberg uncertainty principle can be formulated in terms of the operator $L = -\frac{d^2}{dx^2} + x^2$, which acts on Schwartz space $\mathcal{S}$ by the formula

$$L(f) = -f'' + x^2 f.$$

This operator, sometimes called the *Hermite operator*, is the quantum analogue of the harmonic oscillator. Consider the usual inner product on $\mathcal{S}$ given by

$$(f,g) := \int_{\mathbb{R}} f(x)\overline{g(x)}\, dx, \quad \forall\, f, g \in \mathcal{S}.$$

Exercise 2: Prove that the Heisenberg uncertainty principle implies

$$(Lf, f) \geq (f, f), \ \ \forall\, f \in \mathcal{S}.$$

This is usually denoted by $L \geq 1$. *Hint: Integrate by parts and then use* $2ab \leq a^2 + b^2$.

Remark: The inequality in the above exercise is also very useful in geometry, e.g. it has been used by Witten in his famous analytic proof of the "Morse inequalities" in topology (see [1], Lemma 3.3, for a very readable introduction of Witten's proof).

Annihilation and creation operators: Consider the operators A and A^* defined on $\mathcal{S}$ by

$$A(f) = f' + xf \ \text{ and } \ A^*(f) = -f' + xf.$$

The operators A and A^* are sometimes called the *annihilation and creation operators*, respectively (compare them with $d_{-\phi}$ and δ_ϕ in Proposition 3.2 in [1]).

Exercise 3: Prove that for all $f, g \in \mathcal{S}$, we have

$$(Af, g) = (f, A^*g), \ \ (Af, Af) = (A^*Af, f) \geq 0$$

and

$$A^*A = L - 1.$$

In particular, this again shows that $L \geq 1$.

Witten deformation: Now for each $t \in \mathbb{R}$, consider

$$A_t(f) := f' + txf \ \text{ and } \ A_t^* f := -f' + txf,$$

we call A_t, A_t^* the *Witten deformations* of A and A^*.

Exercise 4: Use the fact that $(A_t^* A_t f, f) \geq 0$ to give another proof of the Heisenberg uncertainty principle which says that whenever $(f, f) = 1$ then

$$(xf, xf) \cdot (f', f') \geq \frac{1}{4}.$$

Hint: think of $(A_t^* A_t f, f)$ *as a quadratic polynomial in* t.

Fourier transform of the Hermite operator: Consider the Hermite operator

$$L_F(f) := -f'' + 4\pi^2 x^2 f, \ \ \forall \ f \in \mathcal{S}$$

associated to the Fourier transform $F(f)(y) = \hat{f}(y) = \int_{\mathbb{R}} f(x) e^{-2\pi i x y} \, dx$.

Exercise 5: Show that $\widehat{L_F(f)} = L_F(\hat{f})$.

Remark: One may think of the above identity as "L_F is a fixed operator of the Fourier transform". Moreover, one may also define the annihilation and creation operator associated to the Fourier transform as

$$A_F(f) = f' + 2\pi x f, \quad A_F^*(f) = -f' + 2\pi x f,$$

(i.e. $A_F = A_{2\pi}$ and $A_F^* = A_{2\pi}^*$). Then one can also prove that

$$\widehat{A_F(f)} = i A_F(\hat{f}), \ \ \widehat{A_F^*(f)} = -i A_F^*(\hat{f}).$$

In particular, if $\hat{f} = f$ then

$$\widehat{A_F(f)} = i A_F(f), \ \ \widehat{A_F^*(f)} = -i A_F^*(f)$$

and iteration gives

$$\widehat{A_F^k(f)} = i^k A_F^k(f), \ \ \widehat{(A_F^*)^k(f)} = (-i)^k (A_F^*)^k(f).$$

In case $f(x) = e^{-\pi x^2}$, the above formula can be used to find the eigenfunctions of the Fourier transform. We will do it in the next exercise set.

REFERENCES

[1] B. Berndtsson, *L^2-estimates for the d-equation and Witten's proof of the Morse inequalities*, Annales de la Faculté des sciences de Toulouse: Mathématiques. Vol. 16. No. 4. 2007.

[2] E. Stein and R. Shakarchi, *Fourier analysis*, Princeton lectures in analysis.