WEEK 7

This exercise set is on the distribution theory (we shall follow a lecture notes [1] by Hasse Carlsson). The first definition is the order of a distribution: let Ω be an open set in $\mathbb{R}^n$, we shall denote by $C_0^\infty(\Omega)$ the space of smooth functions with compact support in Ω.

Definition 1. *A distribution on Ω is a linear functional, say T, on $C_0^\infty(\Omega)$ such that for every compact set K in Ω there exist constant C and N (can depend on K but independent of f)*

$$(1) \qquad |T(f)| \leq C \sum_{|\alpha|\leq N} ||D^\alpha f||_\infty, \quad ||D^\alpha f||_\infty := \sup_{x\in K} |D^\alpha f(x)|,$$

for all $f \in C_0^\infty(\Omega)$ such that $\operatorname{Supp} f \subset K$. If the same N can be used for all K then we say that T has order $\leq N$. The smallest N that can be used is called the order of the distribution.

Remark: We have the following basic facts:

1. Order 0 distribution is also called a *complex measure*;
2. If $D^\alpha T$ is of oder 0 for every α then $T = T_f$ for a smooth function f on Ω;
3. By the partition of unity we know that two distributions are equal iff they are locally equal (see Theorem 2.9 in [1]);
4. If T is a distribution on Ω such that $\operatorname{Supp} T$ is compact in Ω then T has finite order;
5. Let T be an order N distribution. Then T extends uniquely to an functional with the estimate (1) on $C_0^N(\Omega)$. Moreover, $T(f) = 0$ if $f \in C_0^N(\Omega)$ such that $D^\alpha f = 0$ on the support of T for every $|\alpha| \leq N$ (see Theorem 2.13 in [1]).

Exercise 1: Find an infinite order distribution on $(0, 1)$.

Exercise 2: Use Remark 4 and 5 to prove the following statement: let T be a distribution on $\mathbb{R}$ such that $\operatorname{supp} T \subset [0, 1]$ then there exist constants C and N such that

$$(2) \qquad |T(f)| \leq C \sum_{0\leq k\leq N} \sup_{x\in[0,1]} |f^{(k)}(x)|,$$

for every smooth function f on $\mathbb{R}$.

Hint: In the Monday lecture we have proved that for every $\varepsilon > 0$ there exists $C(\varepsilon)$ and $N(\varepsilon)$ such that

$$|T(f)| \leq C(\varepsilon) \sum_{0\leq k\leq N(\varepsilon)} \sup_{x\in[0-\varepsilon,1+\varepsilon]} |f^{(k)}(x)|,$$

for every smooth function f on $\mathbb{R}$. Choose $\varepsilon = 1$, consider $\tilde{f}$ such that

$$\tilde{f}(x) = f(x), \ \forall\, 0 \leq x \leq 1$$

and $\tilde{f}$ is equal to a degree $2N(1)+1$ polynomial

$$P(x) := \sum_{j=0}^{2N(1)+1} c_j \, x^j$$

outside $[0,1]$, where c_j are defined by

$$P^{(k)}(0) = f^{(k)}(0), \;\; P^{(k)}(1) = f^{(k)}(1), \quad 0 \le k \le N(1).$$

Find the formula for c_j and then show that

$$\sum_{0\le k\le N(1)} \sup_{x\in[-1,2]} |\tilde{f}^{(k)}(x)| \le C \sum_{0\le k\le N(1)} \sup_{x\in[0,1]} |f^{(k)}(x)|,$$

where C does not depend on f. Finally use Remark 5 to show that

$$T(f) = T(\tilde{f}),$$

by which (2) follows.

Exercise 3: Show that

$$T(f) := \sum_{n=1}^{\infty} n^{-1} \left(f(1/n) - f(-1/n) \right),$$

defines a distribution of order ≤ 1 on $\mathbb{R}$ and

$$\operatorname{Supp} T = \{0, \pm 1, \pm 1/2, \cdots\}.$$

But the following estimate is not always true

$$|T(f)| \le C \sup_{x\in \operatorname{Supp} T} \left(|f(x)| + |f'(x)| \right), \quad \forall\, f \in C^\infty(\mathbb{R}).$$

Further reading: For a simple proof (using Paley–Wiener theorem and Cauchy integral formula) of the existence of tempered distribution solution for linear differential operator with constant coefficients, see page 66 (Theorem 14.1) in [1].

REFERENCES

[1] H. Carlsson, *Lecture notes on distributions*, http://www.math.chalmers.se/~hasse/distributioner_eng.pdf