WEEK 6

This exercise set is on the complex analysis results used in the proof of the Paley–Wiener theorem. Let us think of a smooth closed curve in $\mathbb{C}$ as a smooth 2π-periodic function $\gamma(t)$ from $\mathbb{R}$ to $\mathbb{C}$, more precisely, the curve associated to γ is the following set

$$\{\gamma(t) : t \in [0, 2\pi)\}$$

with orientation from 0 to 2π. The fundamental theorem in complex analysis is the following:

Theorem 1 (Cauchy integral theorem). *Let f be a holomorphic function on an open set, say U, in $\mathbb{C}$. Assume that inside U we can move a smooth closed curve γ_0 smoothly to another smooth closed curve γ_1. Then*

$$\int_{\gamma_0} f(z)\, dx = \int_{\gamma_1} f(z)\, dx.$$

Exercise 1: Recall that a smooth function on U is said to be holomorphic if it is smooth and satisfies the Cauchy–Riemann equation

$$f_{\bar z} := \partial f/\partial \bar z = 0.$$

Show that if $z = x + iy$, $x, y \in \mathbb{R}$ then we have

$$\partial f/\partial \bar z = \frac{1}{2}(f_x + i f_y),$$

use it to check that $f(z) = z$ is holomorphic and every polynomial of z is holomorphic. Prove the following chain rule

$$df(g(t))/dt = f_z(g(t)) \cdot g'(t), \quad f_z := \partial f/\partial z,$$

where f is holomorphic and g is a smooth function from $t \in (a, b)$ to $\mathbb{C}$.

Remark: In the above Cauchy integral theorem, the precise meaning for "inside U we can move a smooth closed curve γ_0 smoothly to another smooth closed curve γ_1" is the following: there exists a smooth function

$$\gamma : (s, t) \mapsto \gamma(s, t) \in U,$$

on a neighborhood of $(s, t) \in [0, 1] \times \mathbb{R}$ such that each

$$\gamma_s : t \mapsto \gamma(s, t)$$

is 2π-periodic and

$$\gamma(0, t) = \gamma_0(t), \;\; \gamma(1, t) = \gamma_1(t).$$

Use this formulation to do the following exercise.

Exercise 2 [Proof of the Cauchy integral theorem]: Put

$$F(s) = \int_{\gamma_s} f(z)\, dz = \int_0^{2\pi} f(\gamma_s(t))\, d(\gamma_s(t)),$$

show that

$$F'(s) = \int_0^{2\pi} \left(f_z(\gamma_s(t)) \cdot d\gamma_s(t)/ds \cdot \gamma_s'(t) + f(\gamma_s(t)) \cdot d\gamma_s'(t)/ds \right) dt$$

and we can write $f_z(\gamma_s(t)) \cdot d\gamma_s(t)/ds \cdot \gamma_s'(t) + f(\gamma_s(t)) \cdot d\gamma_s'(t)/ds$ as $G'(t)$, where

$$G(t) := f(\gamma_s(t)) \cdot d\gamma_s(t)/ds.$$

Now we have

$$F'(s) = \int_0^{2\pi} G'(t)\, dt = G(2\pi) - G(0).$$

Since G is 2π-periodic we know that $F'(s) \equiv 0$, which gives $F(0) = F(1)$.

Remark: In case γ_1 is just a single point (we write $\gamma_0 \sim 0$ in U) then the Cauchy integral theorem implies

$$\int_{\gamma_0} f(z)\, dz = 0$$

A domain U is said to be *simply connected* if it is possible to move every smooth closed curve smoothly inside U to a single point in U. The Cauchy integral theorem implies

Theorem 2. *Let f be a holomorphic function on a simply connected domain U in $\mathbb{C}$. Then*

$$\int_\gamma f(z)\, dz = 0,$$

for every smooth closed curve in U.

Exercise 3 [Cauchy residue formula]: Let f be a holomorphic function on an open set, say U, in $\mathbb{C}$. Fix $z_0 \in U$. Assume that inside U we can move a smooth closed curve γ smoothly to $\gamma_\varepsilon : t \mapsto z_0 + \varepsilon e^{it}$ for all sufficient small $\varepsilon > 0$. Show that

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz = \lim_{\varepsilon \to 0+} \frac{1}{2\pi i} \int_{\gamma_\varepsilon} \frac{f(z)}{z - z_0}\, dz = f(z_0).$$

Remark: In case f is holomorphic on the disc $U := \{z \in \mathbb{C} : |z| < 3\}$. The Cauchy residue theorem implies that for every $|z| \leq 1$, we have

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(2e^{it})e^{it}}{2e^{it} - z}\, dt.$$

Exercise 4: Use the above formula and

$$\frac{1}{2e^{it} - z} = \frac{1}{2e^{it}}(1 + \omega + \omega^2 + \cdots), \quad \omega := ze^{-it}/2$$

to show that f can be written as a convergence power series of z in $\{|z| \leq 1\}$. Then prove the following "sub-mean inequality"

$$|f(0)|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \,, d\theta, \quad \forall\, 0 \leq r \leq 1.$$

Hint: write $f = \sum c_n z^n$, use the fact that $\{e^{int}\}$ is an orthogonal system.

Exercise 5: Use the sub-mean formula in Exercise 4 to prove the following *maximum principle* for holomorphic functions: If f is holomorphic on a bounded domain U in $\mathbb{C}$ and continuous on the closure, say $\bar{U}$, of U then

$$\sup_{z \in \bar{U}} |f(z)| = \sup_{z \in \partial U} |f(z)|,$$

where ∂U denotes the boundary of U.