WEEK 10–11

In this exercise set we will give a short account of the Fourier analysis theory.

1. Good kernel (from [2]).

Definition 1. *A family of 2π-periodic continuous functions $\{K_n(x)\}_{n=1}^{\infty}$ is said to be a family of good kernels if it satisfies the following properties:*

1) *For all $n \geq 1$, $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1$;*
2) $\sup_{n\geq 1}\int_{-\pi}^{\pi} |K_n(x)|\,dx < \infty$;
3) *For all fixed $\delta > 0$, we have*

$$\lim_{n\to\infty}\int_{\delta\leq|x|\leq\pi} |K_n(x)|\,dx = 0.$$

Remark: By Week 5-1, Exercise 1 and 2, we know that the Fejér kernels

$$F_n(x) := \frac{1}{n}\frac{\sin^2(nx/2)}{\sin^2(x/2)},$$

defines a family of Good kernels. Thus the following exercise can be seen as a generalization of Fejér's theorem.

Exercise 1: Let $\{K_n(x)\}_{n=1}^{\infty}$ be a family of good kernels and f be a 2π-periodic continuous function. Then

$$\lim_{n\to infty}(f\star K_n)(x) = f(x),$$

where $(f\star K_n)(x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)K_n(y)\,dy$.

Remark 1: One may also define a continuous family of good kernels, e g the Poisson kernel $\{P_r(\theta)\}_{0\leq r\leq 1}$. Then the Poisson theorem (Week 5-1, Exercise 5) can also be seen as a standard property of the good kernel.

Remark 2: The main convergence theorem (Theorem 1.1 page 6 in the notes) for the Fourier series only works for C^1 functions (nor for all continuous function). One reason is that the associated family of Dirichlet kernels

$$D_n(x) = \frac{\sin(n+1/2)x}{\sin x/2},$$

is not a family of good kernels (see the following Exercise).

$\star$ **Exercise 2**: Put

$$L_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\sin(n+1/2)x}{\sin x/2}\right|\,dx.$$

Show that $L_N \geq c \ln n$ for some constant $c > 0$. *Hint: Show that*

$$|\frac{\sin(n+1/2)x}{\sin x/2}| \geq c|\frac{\sin(n+1/2)x}{x}|,$$

change of variables, and prove that

$$L_n \geq c \int_{\pi}^{n\pi} \frac{|\sin x|}{|x|}\, dx - C,$$

where C is a constant. Write the integral as a sum $\sum_{k=1}^{n-1} \int_{k\pi}^{(k+1)\pi}$. To conclude, use the fact that

$$\ln n = \ln n - \ln 1 = \int_1^n (\ln x)'\, dx = \int_1^n \frac{1}{x}\, dx \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}, \quad n \geq 2.$$

Remark: One may also define the notion of good kernel on $\mathbb{R}$ (see page 139 in [2] and page 140–141, exercise 9, 10, 14, 16 in page 163–164 in [2]).

2. Poisson summation formula (from [2]).

The original version of the Poisson summation formula (see Theorem 2.2 in the notes) has many applications (see page 25 for its applications in theta relations). One the other hand, in some interesting cases, the original version does not apply. A natural generalization of it is the distribution version of Poisson summation formula (see Theorem 2.7), which is essentially a distribution version of the Fourier series expansion. The distribution version also has many applications, one of them is the elementary solution for $-id/dx + s$ (see page 33). In this section, we shall see how to use the Poisson summation formula to decode relations between kernels on $\mathbb{R}$ and $S^1 = \mathbb{R}/\mathbb{Z}$. From section 1.7.2, we know that

$$H_t(x) := \sum_{k\in\mathbb{Z}} e^{-t\pi k^2} e^{2\pi i k\cdot x}$$

is essentially the heat kernel on S^1, and usually we call

$$\mathcal{H}_t(x) := t^{-\frac{1}{2}} e^{-\frac{\pi x^2}{t}}$$

the heat kernel on $\mathbb{R}$. Since the Fourier transform of $\mathcal{H}_t(t)$ is equal to $e^{-t\pi\gamma^2}$, the Poisson summation formula gives

$$H_t(x) = \sum_{k\in\mathbb{Z}} \mathcal{H}_t(x+k). \tag{1}$$

Thus we get

Fact 1: The heat kernel on $\mathbb{R}/\mathbb{Z}$ is the periodization of the heat kernel on $\mathbb{R}$.

Exercise 3: Recall the definition of the Poisson kernel on $\mathbb{R}/2\pi\mathbb{Z}$

$$P_r(\theta) := \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta} = \frac{1-r^2}{1-2r\cos\theta + r^2}, \quad 0 < r < 1, \quad \theta \in \mathbb{R}.$$

We call

$$\mathcal{P}_y(x) := \frac{1}{\pi}\frac{y}{y^2+x^2}, \ \ y>0, \ x\in\mathbb{R}$$

the Poisson kernel on $\mathbb{R}$ (boundary of the upper half plane). Show that

$$\widehat{\mathcal{P}_y}(\gamma) = e^{-2\pi|\gamma|y}$$

Then use the Poisson summation formula to prove

$$P_r(2\pi x) = \sum_{n\in\mathbb{Z}} \mathcal{P}_y(x+n), \ \ r := e^{-2\pi y}.$$

Remark: The above exercise implies:

Fact 2: The Poisson kernel on $\mathbb{R}/\mathbb{Z}$ is the periodization of the Poisson kernel on $\mathbb{R}$.

One application of the distribution version of the Poisson summation formula is (see (1.5) in page 11, or page 32–33 of the lecture notes)

$$\lim_{N\to\infty} \sum_{|n|\leq N} \frac{1}{n+s} = \pi\cot\pi s, \ \ \forall\ s\ \notin\mathbb{Z}.$$

use it to do the following exercise.

Exercise 4: The Dirichlet kernel on the real line is defined by

$$\int_{-R}^{R} \hat{f}(\xi)e^{2\pi i x\xi}\,d\xi = (f\star\mathcal{D}_R)(x), \text{ so that } \mathcal{D}_R(x) = \widehat{1_{[-R,R]}}(x) = \frac{\sin(2\pi Rx)}{\pi x}.$$

Also the modifies Dirichlet kernel for periodic functions of period 1 is defined by

$$D_R^*(x) := \sum_{|n|\leq R-1} e^{2\pi i n x} + \frac{1}{2}(e^{-2\pi i Rx}+e^{2\pi i Rx}), \ \ R\in\mathbb{Z}_+.$$

Use the above formula for $\pi\cot\pi s$ to prove that

$$\lim_{N\to\infty}\sum_{|n|\leq N} \mathcal{D}_R(x+n) = D_R^*(x), \ \ \forall\ x\in\mathbb{R}, \ \ R\in\mathbb{Z}_+.$$

Remark: The above exercise implies:

Fact 3: The modifies Dirichlet kernel on $\mathbb{R}/\mathbb{Z}$ is the periodization of the Dirichlet kernel on $\mathbb{R}$.

Similar fact also holds for the Fejér kernel (see page 164, Exercise 14 in [2]).

*3. On the zeta function (from [2]).

Exercise 5: The Gamma function is defined for $s>0$ by

$$\Gamma(s) := \int_0^\infty e^{-x}x^{s-1}\,dx.$$

(a) Show that for $s > 0$ the above integral makes sense, that is, that the following two limits exist (hint: use integration by parts for the first limit):

$$\lim_{\delta \to 0+} \int_\delta^1 e^{-x} x^{s-1}\, dx \ \text{ and } \ \lim_{A \to \infty} \int_1^A e^{-x} x^{s-1}\, dx.$$

(b) Prove that $\Gamma(s+1) = s\Gamma(s)$ for all $s > 0$ and conclude that for every integer $n \geq 1$ we have $\Gamma(n+1) = n!$.

(c) Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ (Hint: Use $\int_{\mathbb{R}} e^{-\pi x^2}\, dx = 1$).

Remark: From the definition, we know that $\Gamma(s)$ is in fact holomorphic on $\operatorname{Re} s > 0$. Moreover, $\Gamma(s+1) = s\Gamma(s)$ implies that $\Gamma(s)$ is a meromorphic function on $s \in \mathbb{C}$ with simple poles at $s = 0, -1, -2, \cdots$.

Exercise 6: The zeta function is defined for $s > 1$ by $\zeta(s) := \sum_{n=1}^\infty 1/n^s$. Put

$$\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s),$$

where Γ denotes the Gamma function above. Verify that

$$\xi(s) = \frac{1}{2}\int_0^\infty t^{\frac{s}{2}-1}(\theta(t) - 1)\, dt, \ \ \forall\, s > 1,$$

where the theta function is defined by $\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}$.

Remark: The above exercise implies that $\xi(s)$ is also well defined for all $s \in \mathbb{C}$ such that $\operatorname{Re} s > 0$. Moreover, we can write

$$2\xi(s) = \int_0^1 t^{\frac{s}{2}-1}(\theta(t) - 1)\, dt + \int_1^\infty t^{\frac{s}{2}-1}(\theta(t) - 1)\, dt := I_1 + I_2.$$

It is clear that

$$I_2(s) = \int_1^\infty t^{\frac{s}{2}-1}(\theta(t) - 1)\, dt$$

defines a holomorphic function of $s \in \mathbb{C}$. Moreover, we have

$$I_1(s) = \int_0^1 t^{\frac{s}{2}-1}(\theta(t) - 1)\, dt = \int_0^1 t^{\frac{s}{2}-1}\theta(t)\, dt - \frac{2}{s}, \quad \forall\, s > 0.$$

Notice that the Poisson summation formula implies that

$$\theta(t) = t^{-1/2}\theta(1/t), \ \ \forall\, t > 0.$$

Thus we have (consider $t = 1/x$)

$$\int_0^1 t^{\frac{s}{2}-1}\theta(t)\, dt = \int_1^\infty x^{-2}x^{1-\frac{s}{2}}\theta(1/x)\, dx = \int_1^\infty x^{-1+\frac{1}{2}(1-s)}\theta(x)\, dx.$$

Let us write $\theta(x) = \theta(x) - 1 + 1$, then $\int_1^\infty x^{-1+\frac{1}{2}(1-s)}\, dx = \frac{-2}{1-s}$ gives

$$\int_0^1 t^{\frac{s}{2}-1}\theta(t)\, dt = \int_1^\infty t^{-1+\frac{1}{2}(1-s)}(\theta(t) - 1)\, dt - \frac{2}{1-s}.$$

Now we know that

$$\xi(s)=\frac{1}{2}\int_1^{\infty}\left(t^{-1+\frac{s}{2}}+t^{-1+\frac{1-s}{2}}\right)(\theta(t)-1)\,dt-\frac{1}{s}-\frac{1}{1-s}$$

defines a meromorphic function on $s\in\mathbb{C}$ with simple pole at $s=0,1$ and

$$\xi(s)=\xi(1-s).$$

Together with the meromorphic extension of $\Gamma(s)$, we know that the Riemann zeta function is meromorphic on $s\in\mathbb{C}$ with a simple pole at $s=1$ and $\zeta(-2n)=0$ for $n=1,2,\cdots$. The zeta values $\zeta(2),\zeta(4),\cdots,\zeta(2k),\cdots$ are closely related to the Bernoulli numbers (see page 97–99 and page 166–167 in [2], the Bernoulli numbers are not only interesting in analysis, they also appear naturally in the fundamental Baker–Campbell–Hausdorff formula in Lie group theory, see [1]).

REFERENCES

[1] B. Mielnik and J. Plebański, *Combinatorial approach to Baker–Campbell–Hausdorff exponents*, Annales de l'IHP Physique théorique, **12** (1970), 215–254.
[2] E. Stein and R. Shakarchi, *Fourier analysis*, Princeton lectures in analysis.