TMA 4170 Exercises Week 9 2018

Solution

B&N 5.5. Suppose Φ has compact support and satisfies

$$\Phi(x) = \sum_{k\in\mathbb{Z}} p_k\, \phi(2x-k).$$

Here $p_k = \int_{-\infty}^{\infty} 2\,\Phi(x)\,\overline{\phi(2x-k)}\,dx$.

If $\operatorname{supp}\Phi \subseteq [-a,a]$, then

$p_k = 0$ when $-2a-k \geq a$ or

$2a-k \leq -a$, hence $|k| \geq 3a$.

B&N 5.10. We consider

$$\varphi(x) := \begin{cases} 1-|x|, & |x|\leq 1 \\ 0, & |x|>1. \end{cases}$$

(a) $$\hat{\Phi}(\xi) = 2\,\mathrm{Re}\,\frac{1}{\sqrt{2\pi}}\int_0^1 (1-x)\,e^{-ix\xi}\,dx$$

$$= -2\,\mathrm{Re}\,\frac{1}{\sqrt{2\pi}}\,\frac{1}{i\xi}\int_0^1 e^{-ix\xi}\,dx$$

$$= \frac{2}{\sqrt{2\pi}}\cdot\frac{1}{\xi^2}\,[1-\cos\xi]$$

$$= \frac{4}{\sqrt{2\pi}}\cdot\frac{1}{\xi^2}\cdot\sin^2(\xi/2)$$

(b) We take for granted that

$$\frac{1}{\sin^2(\xi/2)} = \sum_{k\in\mathbb{Z}} \frac{4}{(\xi + 2\pi k)^2}.$$

We differentiate twice. Then the RHS becomes $24 \cdot \sum_{k\in\mathbb{Z}} \frac{1}{(\xi+2\pi k)^4}$. On the other hand,

$$\frac{d}{d\xi}\frac{1}{\sin^2(\xi/2)} = -\frac{\cos(\xi/2)}{\sin^3(\xi/2)}$$

$$\frac{d}{d\xi}\left(-\frac{\cos(\xi/2)}{\sin^3(\xi/2)}\right) = \frac{3}{2}\frac{\cos^2(\xi/2)}{\sin^4(\xi/2)} + \frac{\frac{1}{2}}{\sin^2(\xi/2)}$$

$$= \frac{1}{2}\,\frac{3 - 2\sin^2(\xi/2)}{\sin^4(\xi/2)}.$$

Hence

$$\sum_{k\in\mathbb{Z}} \frac{1}{(\xi+2\pi k)^4} = \underline{\underline{\frac{1}{48} \cdot \frac{3 - 2\sin^2(\xi/2)}{\sin^4(\xi/2)}}}.$$

(c) We set

$$\hat{g}(\xi) := 2 \cdot \sqrt{\frac{2}{\pi}}\; \frac{\sin^2(\xi/2)}{\xi^2\sqrt{1 - \frac{2}{3}\sin^2(\xi/2)}}.$$

Then, by periodicity of the sine function, as well as (b),

$$\sum_k |\hat{g}(\xi + 2\pi k)|^2 = \frac{24}{\pi} \cdot \frac{\sin^4(\xi/2)}{3 - 2\sin^2(\xi/2)}$$

$$\cdot \sum_k \frac{1}{(\xi + 2\pi k)^4} \underset{(b)}{=} \frac{1}{2\pi}.$$

(NOTE: You have to expand

$$\frac{1}{\sqrt{1 - \frac{2}{3}\sin^2(\xi/2)}}$$ to see how g

can be expressed in terms of

integer translates of φ.)

BeN 5.12. We assume $f \in C^1(\mathbb{R})$ with

$|f'(x)| \leq M$, $0 \leq x < 1$. We set

$a_j := f(j/2^n)$ and

$$f_n(x) := \sum_j a_j \, \varphi(2^n x - j),$$

with φ the Haar scaling function.

Let x be an arbitrary point in

$[0, 1]$ and let $j = j(x)$ be such

that $j \leq 2^n x < j+1$. Then

$$|f_n(x) - f(x)| = |a_j - f(x)|$$

$$\leq M \cdot 2^{-n}.$$ This is $\leq \varepsilon$ if

$$2^{-n} \leq \frac{\varepsilon}{M} \iff n \geq \log_2 \frac{M}{\varepsilon}.$$