TMA 4170 Exercises Week 8 2018

Solution

BeN 4.1. We may write

$$f(x) = -\phi(4x) + 4\phi(4x-1) + 2\phi(x-2)$$
$$- 3\phi(x-3).$$

So $(a_0^2, a_1^2, a_2^2, a_3^2) = (-1, 4, 2, -3)$.

Now $b_0^1 = \frac{1}{2}(a_0^2 - a_1^2) = \frac{1}{2}(-1-4) = \underline{\underline{-\frac{5}{2}}}$

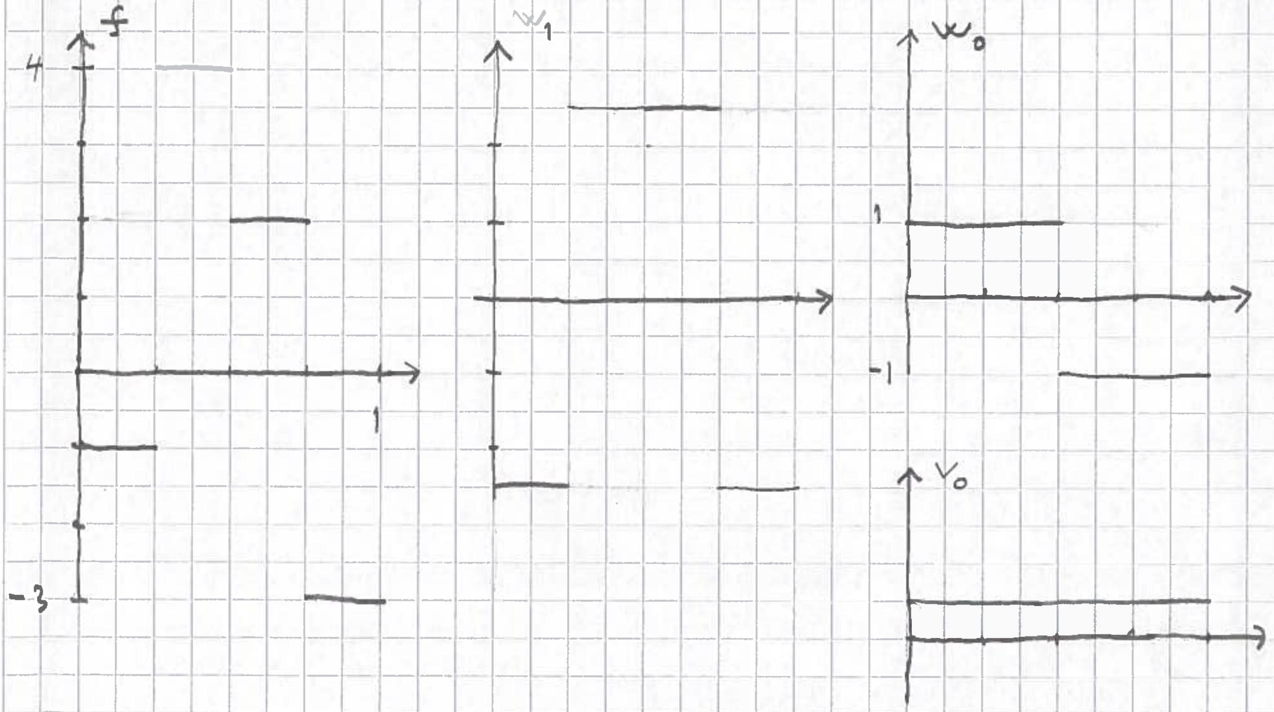
$b_1^1 = \frac{1}{2}(a_2^2 - a_3^2) = \frac{1}{2}(2+3) = \underline{\underline{\frac{5}{2}}}$.

$a_0^1 = \frac{1}{2}(a_0^2 + a_1^2) = \frac{1}{2}(-1+4) = \frac{3}{2}$

$a_1^1 = \frac{1}{2}(a_2^2 + a_3^2) = \frac{1}{2}(2-3) = -\frac{1}{2}$. Then

$b_0^0 = \frac{1}{2}(a_0^1 - a_1^1) = \frac{1}{2}(\frac{3}{2} + \frac{1}{2}) = \underline{\underline{1}}$

$a_0^0 = \frac{1}{2}(a_0^1 + a_1^1) = \frac{1}{2}(\frac{3}{2} - \frac{1}{2}) = \underline{\underline{\frac{1}{2}}}$



B&N 4.3 Suppose $\{e_1, \dots, e_m\}$ is an ONB for A and $\{f_1, \dots, f_n\}$ is an ONB for B. If $A \perp B$, then $\{e_1, \dots, e_m, f_1, \dots, f_n\}$ is an ONB for $A \oplus B$, and hence

$$\dim(A \oplus B) = m + n.$$

In general, we have

$$\max(\dim A, \dim B) \leq \dim(A + B) \leq \dim A + \dim B.$$

B&N 4.4. (a) We set

$$V_n := \operatorname{span}\{\phi(2^n x), \phi(2^n x - 1), \dots, \phi(2^n x - 2^n + 1)\}$$

$$W_n := \operatorname{span}\{\psi(2^n x), \psi(2^n x - 1), \dots, \psi(2^n x - 2^n + 1)\}.$$

Hence $\dim V_n = \dim W_n = 2^n$.

(b) $V_n = W_{n-1} \oplus W_{n-2} \oplus \cdots \oplus W_0 \oplus V_0$.

Hence, by problem 4.3,

$$\dim V_n = 2^{n-1} + 2^{n-2} + \dots + 1 + 1$$

$$= \frac{2^n - 1}{2 - 1} + 1 = 2^n - 1 + 1 = \underline{\underline{2^n}},$$

in agreement with (a).

B & N 4.5. Given

$$f(x) = \sum_k a_k \, \phi(2x - k).$$

We assume that $f(x) \perp \phi(x - \ell)$ for every ℓ in $\mathbb{Z}$. This implies:

$$0 = \int_{-\infty}^{\infty} f(x)\, \phi(x - \ell)\, dx$$

$$= \int_{-\infty}^{\infty} f(x + \ell)\, \phi(x)\, dx = \int_0^1 f(x + \ell)\, dx$$

$$= \sum_k a_k \int_0^1 \phi(2x - k + 2\ell)\, dx$$

$$= \frac{1}{2}(a_{2\ell} + a_{2\ell+1}).$$ Hence

$$a_{2\ell} = - a_{2\ell+1}, \quad \ell \in \mathbb{Z}.$$

Since $\psi(x) = \phi(2x) - \phi(2x - 1)$, this gives

$$f(x) = \sum_\ell a_{2\ell}\, \psi(x - \ell).$$