TMA 4170 Exercises Week 7 2018

Solution

B&N 3.2

- $y \in S_n$ and $z_k = y_{k+1}$

  $\Rightarrow \mathcal{F}[z]_j = w^j \mathcal{F}[y]_j$

Proof: $\mathcal{F}[z]_j = \sum_{l=0}^{n-1} z_l \overline{w}^{lj} = \sum_{l=0}^{n-1} y_{l+1} \overline{w}^{lj}$

$$= \sum_{l=1}^{n-1} y_l \overline{w}^{j(l-1)} + y_0 \overline{w}^{j(n-1)} = w^j \sum_{l=0}^{n-1} y_l \overline{w}^{jl}.$$

- $[y * z]_k := \sum_{j=0}^{n-1} y_j z_{k-j}$ is in S_n because

$$[y * z]_{k+n} = \sum_{j=0}^{n-1} y_j z_{k+n-j} = \sum_{j=0}^{n-1} y_j z_{k-j}.$$

- $\mathcal{F}[y * z]_k = \sum_{l=0}^{n-1} \sum_{j=0}^{n-1} y_j z_{l-j} \overline{w}^{lk}$

$$= \sum_{l=0}^{n-1} \sum_{j=0}^{n-1} y_j \overline{w}^{kj} z_{l-j} \overline{w}^{k(l-j)}$$

$$= \sum_{j=0}^{n-1} y_j \overline{w}^{kj} \sum_{l=0}^{n-1} z_l \overline{w}^{kl}.$$

- Suppose $y \in S_n$ is real valued. Then

$$\hat{y}_{n-k} = \sum_{j=0}^{n-1} y_j \overline{w}^{j(n-k)} = \sum_{j=0}^{n-1} y_j \overline{w}^{-jk}$$

$$= \sum_{j=0}^{n-1} \overline{y}_j \overline{\overline{w}^{jk}} = \overline{\hat{y}_k}.$$

B & N 3.3. We will see later that $\hat{\delta}(\xi) \equiv 1$. Hence if $au'' + bu' + cu = f_0 \delta$, then $(-a\xi^2 + ib\xi + c)\hat{u} = f_0$, and so

$$u(t) = \mathcal{F}^{-1}\left(\frac{f_0}{-a\xi^2 + ib\xi + c}\right)(t).$$

Writing $\sin \omega t = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$, we get

$$\int_0^\infty \sin \omega t \, e^{-\mu t - i\xi t} \, dt$$

$$= \frac{1}{2i}\left(\frac{1}{i(\omega - \xi) - \mu} - \frac{1}{-i(\omega + \xi) - \mu}\right)$$

$$= \frac{1}{2i} \frac{-2i\omega}{\omega^2 - \xi^2 + \mu^2 - 2i\mu\xi}.$$

$$= \frac{\omega a}{-a\xi^2 + ib\xi + c}$$

since $\omega = \frac{\sqrt{4ac - b^2}}{2a}$, $\mu = \frac{b}{2a}$.

B & N 3.10 Inserting $\frac{u_k - u_{k-1}}{h}$ for u' and $\frac{u_{k+1} + u_{k-1} - 2u_k}{h^2}$ for u'' in

$$au'' + bu' + cu = f,$$ we get

$$\frac{a u_{k+1}}{h^2} + \left(\frac{a}{h^2} - \frac{b}{h}\right) u_{k-1} + \left(-\frac{2a}{h^2} + \frac{b}{h} + c\right) u_k$$

$= f_k$. Hence, setting

$\beta = ch^2 + bh - 2a$ and $\gamma = a - bh$, we get

$$a u_{k+1} + \beta u_k + \gamma u_{k-1} = h^2 f_k . \quad (**)$$

Bew 3.11. We consider $az^2 + \beta z + \gamma = 0$, which has solutions

$$z = \frac{-\beta \pm \sqrt{\beta^2 - 4a\gamma}}{2a} .$$

If $\beta^2 \le 4a\gamma$, then solutions on $\mathbb{T}$ occur $\iff \frac{\gamma}{a} = 1$, but this is impossible since $\gamma = a - bh < a$.

If $\beta^2 > 4a\gamma$, then we must have either $a + \beta + \gamma = 0$ or $a - \beta + \gamma = 0$.

Now $a + \beta + \gamma = ch^2 > 0$ and

$$a - \beta + \gamma \le 2\sqrt{a\gamma} - \beta$$

$$< \beta - \beta = 0 .$$

B & N 3.12.

We take the DFT of (**) and use the first part of 3.2:

$$(\alpha \omega^{j} + \beta + \gamma \omega^{-j})\, \hat{u}_j = h^2 \hat{f}_j .$$

The result follows since $\alpha \omega^{j} + \beta + \gamma \omega^{-j}$ by the preceding exercise.