TMA 4170 Exercises Week 6 2018

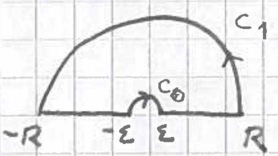
Solution

Problems on the Paley-Wiener theorem

(1) Clearly,

$$\int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x}\, dx = \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_{\varepsilon < |x| < R} \operatorname{Im} \frac{e^{i\pi x}}{\pi x}\, dx .$$

The two segments $\varepsilon < |x| < R$ are part of the contour suggested in the problem:



We consider first

$$\int_{C_0} \frac{e^{i\pi z}}{\pi z}\, dz = -i + O(\varepsilon), \quad \varepsilon \to 0,$$

by explicit computation.

On the other hand,

$$\left| \int_{C_1} \frac{e^{i\pi z}}{\pi z}\, dz \right| \le \frac{1}{\pi} \int_0^{\pi} e^{-\pi R \sin\theta}\, d\theta$$

$$\le \frac{2}{\pi} \int_0^{\pi} e^{-2R\theta}\, d\theta \le \frac{1}{\pi R} .$$

Using the initial observation and letting $\varepsilon \to 0$ and $R \to \infty$, we get

$$\int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x}\, dx = 1 .$$

(2) Integrating by parts, we get

$$\int_{-\pi}^{\pi} \frac{\sin^2 \pi x}{\pi^2 x^2} = \left[- \frac{\sin^2 \pi x}{\pi^2 x} \right]_{-\infty}^{\infty}$$

$$+ \int_{-\infty}^{\infty} \frac{\sin 2\pi x}{\pi x} dx = \int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x} dx = 1.$$

(3) We notice that

$$\frac{1}{(x-m)} \cdot \frac{1}{(x-n)} = \frac{1}{(n-m)} \left(\frac{1}{x-n} - \frac{1}{x-m} \right).$$

Now $\displaystyle\int_{-\infty}^{\infty} \frac{\sin \pi (x-n)}{(x-n)} \cdot \sin \pi (x-m) \, dx$

$$= \int_{-\infty}^{\infty} \frac{\sin^2 \pi x}{x} \cdot (-1)^{n-m} dx = 0$$

because the integrand is an odd function. Hence

$$\int_{-\infty}^{\infty} \frac{\sin \pi (x-n)}{\pi (x-n)} \cdot \frac{\sin \pi (x-m)}{\pi (x-m)} dx = 0.$$

(4) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x} e^{-ix\xi} dx$

$$= \int_{-\infty}^{\infty} \frac{\sin \pi x \cos x\xi}{\pi x} dx$$

$$= \frac{1}{2} \left(\int_{-\infty}^{\infty} \frac{\sin (\pi + \xi) x}{\pi x} dx + \int_{-\infty}^{\infty} \frac{\sin (\pi - \xi) x}{dx} dx \right).$$

Here both integrals are 1 if $|\xi| < \pi$.
If $|\xi| = \pi$, then one of them is 0.
If $|\xi| > \pi$, then they are ± 1.

B & N 2.14

(a) Suppose $\hat{f}(\xi) = 0$ for $|\xi| \geq \Omega$.

The fact that

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} c_{-n}\, e^{-in\pi\xi/(a\Omega)}, \quad |\xi| \leq a\Omega,$$

is immediate.

(b) Here we could choose <u>any</u> function g_a with the property that

$$\hat{g}_a(\xi) = \begin{cases} 1, & |\xi| \leq \Omega \\ 0, & |\xi| \geq a\Omega \end{cases};$$

we could for example choose $\hat{g}_a$ to be a C^∞ function, and hence we could choose g_a to be a Schwartz function. (I omit the explicit computation required in the problem - it is straightforward.)

(c) Since $\hat{f}(\xi) = \hat{f}(\xi)\,\hat{g}_a(\xi)$

$$= \sum_{n=-\infty}^{\infty} c_{-n}\, e^{-in\pi\xi/(a\Omega)}\, \hat{g}_a(\xi)$$

for <u>all</u> ξ, we get

$$f(t) = \sum_{n=-\infty}^{\infty} c_{-n}\, g_a\!\left(t - \frac{n\pi}{a\Omega}\right)$$

by Thm. 2.6.6.