TMA 4170 Exercises Week 5 2018

Solution

Problems on $\mathcal{S}$ and the Poisson formula

(1) Suppose f is in $\mathcal{S}$ and write

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\xi x}\, dx .$$

By the rapid decay of f, we may differentiate $\hat{f}(\xi)$:

$$\frac{d^m}{d\xi^m} \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)(-ix)^m e^{-i\xi x}\, dx$$

Similarly, by integrating by parts n times, we get

$$(i\xi)^n \frac{d^m}{d\xi^m} \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d^n}{dx^n}\left[f(x)(-ix)^m\right] e^{-i\xi x}\, dx.$$

Since f is in $\mathcal{S}$, the right-hand side is bounded for all m, n, and hence $\hat{f}$ is also in $\mathcal{S}$. $\mathcal{F}$ maps $\mathcal{S}$ onto itself by the inversion formula and the preceding argument.

(2) We consider

$$f(x) := \begin{cases} 0, & x \le 0 \\ \exp(-x^{-1} e^{1/(x-1)}), & 0 < x < 1 \\ 1, & x \ge 1. \end{cases}$$

We observe that

$$\lim_{x \to 0^+} f(x) = 0 \quad \text{and} \quad \lim_{x \to 1^-} f(x) = 1$$

We wish to show that f is in $C^\infty(\mathbb{R})$. To this end, it suffices to verify that

$$f^{(n)}(0) = f^{(n)}(1) = 0.$$

We observe that for $0 < x < 1$

$$f^{(n-1)}(x) = e^{-x^{-1} e^{1/(x-1)}} \cdot g_n(x)$$

where we first write

$$g_n(x) = x^{-2(n-1)} a_{2(n-1)}(x) + \dots + x^{-1} a_1(x),$$

where the a_j are bounded at 0.

Hence $$\lim_{x \to 0^+} \frac{f^{(n-1)}(x)}{x} = 0$$

because $\lim_{x \to 0^+} x^{-k} f(x) = 0$ for all $k > 0$. Around 1 we write $g_n(x)$ as a polynomial in $\frac{1}{x-1}$ and $e^{\frac{1}{x-1}}$ and use that

$$\lim_{x \to 0^-} \frac{1}{(x-1)^k} e^{\frac{1}{x-1}} = 0$$

for all k to conclude that

$$\lim_{x \to 0^-} \frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x-1} = 0$$

for all $n \geq 1$.

Clearly, $f(\lambda(x-a)) - f(\lambda(x-b))$ will be in $\mathcal{S}$ since it is C^∞ and compactly supported. When $\lambda \to \infty$, this function will approximate $\chi_{[a,b]}(x)$.

(3) The function $\sum_{n=-\infty}^{\infty} f(x+n)$ is a C^∞ function by the smoothness and decay of f. It is clearly 1-periodic by a trivial re-indexation of the series. The kth Fourier coefficient of this function is

$$\int_0^1 \left[\sum_{n=-\infty}^{\infty} f(x+n)\right] e^{-2\pi i k x}\, dx$$

$$= \sum_{n=-\infty}^{\infty} \left(\int_n^{n+1} f(x) e^{-2\pi i k x}\, dx\right)$$

$$= \hat{f}(k).$$ Hence

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi i x k}.$$

(4) We rewrite the result of (3) as

$$\sum_{n=-\infty}^{\infty} f(x+nT) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \hat{f}\left(\frac{k}{T}\right) e^{2\pi i k x/T}.$$

Here the LHS $\to f(x)$ when $T \to \infty$, while the RHS is a Riemann sum with mesh $1/T$ for the integral $\int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x}$.

(5) We recall that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-ix\xi} \, dx = e^{-\xi^2/2}.$$

Using the present definition of the Fourier transform and this relation, we get $f(x) = e^{-\pi t x^2}$ has

$$\hat{f}(\xi) = t^{-\frac{1}{2}} e^{-\pi \xi^2/t}.$$

Jacobi's identity now follows from the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k).$$

B & N 2.4. Suppose f in $L^2(\mathbb{R})$ is real-valued and even. Then

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f(x) \cos(x\xi) \, dx$$

because $f(x) \sin(x\xi)$ is an odd function. On the other hand, if f is real-valued and odd, then

$$\hat{f}(\xi) = \frac{-2i}{\sqrt{2\pi}} \int_0^\infty f(x) \sin(x\xi)\, dx$$

because $f(x)\cos(x\xi)$ is an odd function.

B&N 2,5. Consider the function

$$\phi(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & \text{otherwise}. \end{cases}$$

Then

$$\phi * \phi(x) = \int_0^1 \chi_{[0,1]}(x-y)\, dx$$

$$= \begin{cases} \int_0^x dx, & 0 \le x \le 1 \\ \int_{x-1}^1 dx, & 1 < x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 < x \le 2 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1-|x-1|, & 0 \le x < 2 \\ 0 & \text{otherwise}. \end{cases}$$