TMA4170 Exercises Week 4 2018

Solution

Problems on $D_n(x)$ and $F_n(x)$.

(1) Since $D_n(x) = \frac{1}{2\pi} \sum_{|m| \le n} e^{imx}$ and $\int_{-\pi}^{\pi} e^{imx}\, dx = 0$ when $m \ne 0$, we have $\int_{-\pi}^{\pi} D_n(x)\, dx = 1$. This also implies that $\int_{-\pi}^{\pi} F_n(x)\, dx = 1$ since $F_n = \frac{1}{n} \sum_{m=0}^{n-1} D_m$.

(2) We see that

$$F_n(x) = \frac{1}{2\pi n \sin(\frac{x}{2})} \operatorname{Im} \sum_{m=0}^{n-1} e^{i(n+1/2)x}$$

$$= \frac{1}{2\pi n \sin(\frac{x}{2})} \operatorname{Im} e^{i\frac{x}{2}} \left(\frac{1 - e^{inx}}{1 - e^{ix}} \right)$$

$$= \frac{1}{2\pi n \sin(\frac{x}{2})} \cdot \frac{1}{2} \frac{1 - \cos nx}{\sin(\frac{x}{2})}$$

$$= \frac{\sin^2 \frac{nx}{2}}{2\pi n \sin^2 \frac{x}{2}}.$$

(3) On the interval $\left[\frac{(k-\frac{1}{2})\pi}{2(n+\frac{1}{2})}, \frac{(k+\frac{1}{2})\pi}{2(n+\frac{1}{2})}\right]$,

$$|D_n(x)| \ge C \frac{n+\frac{1}{2}}{k-\frac{1}{2}}, \quad k = 1, \dots, n-1.$$

Restricting the integral to these intervals and noting that each of them has length $\frac{\pi}{2} \cdot \frac{1}{(n+\frac{1}{2})}$, we get

$$\int_{-\pi}^{\pi} |D_n(x)\,dx| \geq c \sum_{k=1}^{n-1} \frac{1}{k-\frac{1}{2}} \geq c \log n.$$

(4) We use the result in (2):

$$\int_{\varepsilon \leq |x| \leq \pi} F_n(x)\,dx \leq \frac{1}{\pi n} \int_{\varepsilon}^{\pi} \frac{\pi^2}{x^2}\,dx$$

$$= \frac{\pi}{\varepsilon n},$$ and the result follows

since $\int_{-\pi}^{\pi} F_n(x)\,dx = 1$.

(5) We write

$$|f(x) - F_n * f(x)|$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x) - f(y))\,F_n(x-y)\,dy$$

$$\leq \frac{1}{2\pi} \int_{|x-y| \leq \varepsilon} |f(x) - f(y)|\,F_n(x-y)\,dy$$

$$+ \frac{1}{2\pi} \int_{\varepsilon \leq |y| \leq \pi} 2 \cdot \|f\|_\infty\,F_n(y)\,dy$$

$$\leq \sup_{|\delta| \leq \varepsilon} |f(x) - f(x-\delta)|$$

$$+ \frac{C \cdot \|f\|_\infty}{\max(1, \varepsilon n)}.$$

We now choose ε small enough to make the first term uniformly small in x and then n large enough to make the second term as small as we please.

(6) Suppose $\varepsilon > 0$ and f in $L^1(\mathbb{T})$ are given. Pick g in $C(\mathbb{T})$ such that $\| f - g \|_1 < \varepsilon/3$. Then

$$\| f - f_\delta \|_1 = \| f - g + g - g_\delta + g_\delta - f_\delta \|_1$$

$$\leq 2 \| f - g \|_1 + \| g - g_\delta \|_1 < \frac{2\varepsilon}{3} + 2\pi \| g - g_\delta \|_\infty .$$

Thus to get $\| f - f_\delta \|_1 < \varepsilon$ it suffices to choose η such that $\| g - g_\delta \|_\infty < \frac{\varepsilon}{6\pi}$ when $|\delta| \leq \eta$.

(7) We make the same splitting as in (5); we integrate the inequality and use (6).

<u>B & N 1.1.</u> $f(x) = x^2$. We get

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x^2 \, dx = \frac{1}{\sqrt{2\pi}} \cdot \frac{2\pi^3}{3} .$$

$$\hat{f}(n) \underset{n \neq 0}{=} \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{in} \int_{-\pi}^{\pi} x \, e^{-inx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{4\pi}{n^2} \cdot (-1)^n .$$ Hence

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \, , \quad |x| \leq \pi .$$

Convergence at $x = \pi$ is clear because $(f(x) - f(\pi))/(x - \pi)$ is integrable. Alternatively, apply

Thm. 1.28. Plugging in $x = \pi$, we get the desired result

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

(which we earlier saw could be obtained from Parseval's identity).