TMA4170 Exercises Week 16

Solution

(1) We see that the function

$$g(x) := D e^{-\pi x^2} = -2\pi x \, e^{-\pi x^2} \quad \text{satisfies:}$$

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} D e^{-\pi x^2} e^{-2\pi i x \xi} \, dx$$

$$= \int_{-\infty}^{\infty} e^{-\pi x^2} D e^{-2\pi i x \xi} \, dx$$

$$= -2\pi i \xi \, e^{-\pi \xi^2} = i \, g(\xi).$$

Elaborating this idea, we are led to the Hermite functions that diagonalize the Fourier transform.

Since $\mathcal{F} f = \mathcal{F}^{-1} f$ for even functions f and $\mathcal{F} f = -\mathcal{F}^{-1} f$ for odd functions f, decomposing $f = g + h$, g even, h odd, we get

$$\mathcal{F}^2 f = \mathcal{F}\mathcal{F}^{-1} g - \mathcal{F}\mathcal{F}^{-1} h = g - h.$$

Hence $\mathcal{F}^4 f = f$. This shows that the constant must be a 4th root of unity.

(2) We set

$$F_R(x) := R\left(\frac{\sin \pi R x}{\pi R x}\right)^2.$$

We begin by observing that $\hat{F}_R(\xi) = \hat{F}_1(\xi/R)$. We know from the problems from week 6 that

$$\int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x} e^{-2\pi i \xi x} \, dx = \chi_{[-\frac{1}{2},\frac{1}{2}]}(\xi).$$

Hence $\hat{F}_1(\xi) = \chi_{[-\frac{1}{2},\frac{1}{2}]} * \chi_{[-\frac{1}{2},\frac{1}{2}]}(\xi)$

$$= \begin{cases} 1-|\xi|, & |\xi| \le 1 \\ 0, & \text{otherwise}. \end{cases}$$

We have $F_R(x) \ge 0$, $\int F_R = 1$. Also:

$$\int_{|x|>\delta} F_R(x)\, dx \le 2 \int_\delta^\infty \frac{dx}{\pi^2 R x^2} = \frac{2}{\pi^2 R \delta} \to 0$$

when $R \to \infty$. Hence F_R is "good".

(3) $$\sum_{n=-\infty}^{\infty} F_N(x+n) = \sum_{n=-\infty}^{\infty} \hat{F}_N(n) e^{2\pi i n x}$$

$$= \sum_{|n| \le N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x}$$

which we recognize as the Fejér kernel on $\mathbb{T}$. (See Exercises week 4).

(4) We integrate by parts;

$$\Gamma(s+1) = \int_0^\infty x^s e^{-x}\, dx$$

$$= \left[-x^s e^{-x}\right]_0^\infty + s\int_0^\infty x^{s-1} e^{-x}\, dx$$

$$= s\,\Gamma(s).$$

Now

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty x^{-\frac{1}{2}} e^{-x}\, dx$$

$$\underset{x=u^2}{=} 2\int_0^\infty e^{-u^2}\, du = \sqrt{\pi}.$$

By the functional equation, we get

$$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$