TMA 4170 Exercises Week 12

Solution

(1) Making the change of variables $u = t 2\pi$ and setting

$$\tilde{x}(u) := x\left(\frac{u}{2\pi}\right), \quad \tilde{y}(u) := y\left(\frac{u}{2\pi}\right),$$

we get

$$x(t)\, y'(t)\, dt = \tilde{x}(u) \cdot \tilde{y}'(u)\, du \quad \text{and}$$

$$\sqrt{x'(t)^2 + y'(t)^2}\, dt = \sqrt{\tilde{x}'(u)^2 + \tilde{y}'(u)}\, du.$$

This solves the first part. Since

$$\int_0^{2\pi} (x(t) - a)\, y'(t)\, dt = \int_0^{2\pi} x(t)\, y'(t)\, dt$$

by periodicity of $y(t)$, we may assume that $\int_0^{2\pi} x(t)\, dt = 0$.

If we now set $x = f$ and

$y(t) = \int_0^t f(\tau)\, d\tau$ in (1), then we get

$$\int_0^{2\pi} |f(t)|^2\, dt \le \frac{1}{4\pi} \left(\int_0^{2\pi} \sqrt{f'(t)^2 + f(t)^2}\, dt \right)^2$$

$$\le \frac{1}{2} \int_0^{2\pi} f'(t)^2\, dt + \frac{1}{2} \int_0^{2\pi} f(t)\, dt,$$

using Cauchy-Schwarz in the last step. Wirtinger's inequality follows.

(2) The curve Γ of length L is parametrized by $x = x(t)$, $y = y(t)$, $0 \le t \le 2\pi$, such that

$$|x'(t)|^2 + |y'(t)|^2 = L^2/(2\pi)^2 .$$

The isoperimetric inequality will follow if we can show that

$$I := \frac{1}{2} \left| \int_0^{2\pi} (x(t)\, y'(t) - y(t)\, x'(t)\, dt \right| \le \frac{L^2}{4\pi} .$$

Since $|ab + cd| \le \sqrt{a^2 + c^2} \cdot \sqrt{b^2 + d^2}$, we get, again by Cauchy-Schwarz + Wirtinger:

$$I \le \frac{1}{2} \int_0^{2\pi} \sqrt{x(t)^2 + y(t)^2} \cdot \frac{L}{2\pi}\, dt$$

$$\le \frac{L}{4\pi} \sqrt{2\pi} \left(\int_0^{2\pi} (x(t)^2 + y(t)^2)\, dt \right)^{1/2}$$

$$\le \frac{L}{4\pi} \cdot \frac{L}{2\pi} \cdot \sqrt{2\pi} \cdot \sqrt{2\pi} = \frac{L^2}{4\pi} .$$

(3) Suppose $f \in C^1([0, N])$. Then

$$\int_m^{m+1} f(x)\, dx = f(m+1) - \int_m^{m+1} (x - m)\, f'(x)\, dx ,$$

$m = 0, \dots, N$. Summing, we get

$$\sum_{m=1}^{N} f(m) = \int_0^N f(x)\, dx + \int_0^N (x - [x])\, f'(x)\, dx .$$

(4) To show that $\{an^\sigma\}$ is equidistributed in $[0,1)$ we need to show that

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k a n^\sigma} \to 0 \quad \text{for } k=1,2,\ldots.$$

By the preceding problem,

$$\sum_{n=1}^{N} e^{2\pi i k a n^\sigma} = \int_0^N e^{2\pi i k a x^\sigma}\,dx$$

$$+\ 2\pi i k a \int_0^N \sigma x^{\sigma-1}\,(x-[x])\,dx$$

$=: I + R$, say. Clearly,

$|R| \le C\cdot N^\sigma$ since $|x-[x]| \le 1$.

Also, by integration by parts,

$$|I| \le \frac{1}{2\pi k a \sigma}\left|\int_0^N x^{1-\sigma}\frac{d}{dx}e^{2\pi i k a x^\sigma}\,dx\right|$$

$$\le C_1 N^{1-\sigma} + C_2\int_0^N x^{-\sigma}\,dx$$

$$= C_3 N^{1-\sigma}.$$

Since $0<\sigma<1$, the result follows.