TMA 4170 Exercises Week 12

Solution

(1) We begin by observing that

$$[-\tfrac{1}{2}, \tfrac{1}{2}]^d \supseteq B(0, \tfrac{1}{2}) := \{x : |x| \le \tfrac{1}{2}\}.$$

Since $\cos t \ge 1 - \frac{t^2}{2}$, we get

$$1 - f_d(x) \le \frac{2\pi^2}{d} \cdot |x|^2 . \quad \text{Hence}$$

$$\int_{\mathbb{T}^d} \frac{dx}{1 - f_d(x)} \ge \frac{2d}{2\pi^2} \cdot \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^{\frac{1}{2}} r^{d-3}\, dr$$

which is $+\infty$ when $d = 2$. On the other hand, when $x \notin B(0, \frac{1}{2})$, there is j such that $|x_j| \ge \frac{1}{2\sqrt{d}}$.

Thus

$$1 - f_d(x) \ge \frac{1}{d}\left(1 - \cos \frac{1}{2\sqrt{d}}\right) = c_d, \text{ say, so}$$

$$\int_{\mathbb{T}^d \setminus B(0, \frac{1}{2})} \frac{dx}{1 - f_d(x)} \le \frac{1}{c_d} .$$

Since $\cos t \le 1 - \frac{t^2}{\pi^2}$, $|t| \le \pi$,

$$\int_{B(0,\frac{1}{2})} \frac{dx}{1 - f_d(x)} \le \frac{4d \cdot 2\pi^{d/2}}{\Gamma(d/2)} \int_0^{\frac{1}{2}} r^{d-3}\, dr$$

$< \infty$ when $d \ge 3$.

(2) We assume that f_n are positive and g is positive and bounded. Moreover, we have

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(x) k(x) dx = \int_{-\infty}^{\infty} g(x) k(x) dx$$

whenever $k \in S$. We now choose k^+, k^- in S such that both k^+ and k^- take values in $[0, 1]$ and

$$k^-(x) \leq \chi_{[a,b]}(x) \leq k^+(x).$$

Then

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(x) k^-(x) dx \leq \liminf_{n \to \infty} \int_a^b f_n(x) dx$$

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(x) k^+(x) dx \geq \limsup_{n \to \infty} \int_a^b f_n(x) dx.$$

We may choose k^- and k^+ such that the two integrals

$$\int_{-\infty}^{\infty} g(x) k^{\mp}(x) dx$$

come as close as we wish to

$$\int_a^b g(x) dx.$$

Hence the liminf and limsup above must be the same, so that

$$\lim_{n \to \infty} \int_a^b f_n(x) dx = \int_a^b g(x) dx.$$

(3) By the binomial formula,

$$(1+e^{2\pi i x})^n = \sum_{j=0}^{n} \binom{n}{j} e^{2\pi i x j}.$$

Hence

$$\int_0^1 |1+e^{2\pi i x}|^{2n}\,dx = \sum_{j=0}^{n} \binom{n}{j}^2$$

by Parseval. On the other hand,

$$|1+e^{2\pi i x}|^2 = (2\cos \pi x)^2$$

$$= (e^{\pi i x} + e^{-\pi i x})^2.$$ Consequently,

$$|1+e^{2\pi i x}|^{2n} = e^{2\pi i n x}(1+e^{-2\pi i x})^{2n}$$

$$= e^{2\pi i n x} \sum_{j=0}^{2n} \binom{2n}{j} e^{-2\pi i j x}.$$ Hence

$$\int_0^1 |1+e^{2\pi i x}|^{2n}\,dx = \binom{2n}{n}.$$