TMA4170 Exercises Week 11

Solution

(1) We seek $M(\xi)$ satisfying

$$(*) \quad M(\xi) + M(\xi + \pi) = 1,$$

We start from

$$1 = \left(\cos^2 \frac{\xi}{2} + \sin^2 \frac{\xi}{2}\right)^{2m+1}$$

$$= \sum_{j=0}^{m} \binom{2m+1}{j} \Big(\cos^{2(2m+1-j)} \xi/2 \, \sin^{2j} \xi/2$$

$$+ \cos^{2j} \xi/2 \, \sin^{2(2m+1-j)} \xi/2 \Big).$$

Now $\cos^{2k} \xi = \sin^{2k}(\xi + \frac{\pi}{2})$ and

$\sin^{2k} \xi = \cos^{2k}(\xi + \frac{\pi}{2})$, so we see that

$$M(\xi) = \sum_{j=0}^{m} \binom{2m+1}{j} \cos^{4m+2-2j} \xi/2 \, \sin^{2j} \xi/2$$

satisfies $(*)$.

(2) We may write

$$M(\xi) = |1 + e^{-i\xi}|^{2m+2}$$

$$\cdot \sum_{j=0}^{m} \binom{2m+1}{j} \cos^{2m-2j} \frac{\xi}{2} \sin^{2j} \frac{\xi}{2}.$$

The latter sum can be expressed as a polynomial in $\cos \xi$.

We may then write (since the polynomial is nonnegative):

$$\sum_{j=0}^{n} \binom{2n+1}{j} \cos^{2n-2j} \frac{\xi}{2} \sin^{2j} \frac{\xi}{2}$$

$$= C \prod_{j=1}^{n} (\cos \xi - z_j)(\cos \xi - \overline{z}_j) .$$

Now writing $\cos \xi = \frac{1}{2}\left(z + \frac{1}{z}\right)$ when $z \in \mathbb{T}$, we see that the polynomials $z^2 - 2 z_j z + 1$ and $z^2 - 2 \overline{z}_j z + 1$ will vanish at four points, say $w_j, \frac{1}{w_j}, \overline{w_j}, \frac{1}{\overline{w_j}}$.

Hence

$$\sum_{j=0}^{n} \binom{2n+1}{j} \cos^{2n-2j} \frac{\xi}{2} \sin^{2j} \frac{\xi}{2}$$

$$= C' \prod_{j=1}^{n} \left| (e^{-i\xi} - w_j)(e^{-i\xi} - \overline{w_j})(e^{-i\xi} - \frac{1}{w_j})(e^{-i\xi} - \frac{1}{\overline{w_j}}) \right|$$

$$= C' \left| \prod_{j=1}^{n} |w_j|^{-2} (e^{-i\xi} - w_j)(e^{-i\xi} - \overline{w_j}) \right|^2 .$$

This gives the desired polynomial:

$$m_0(\xi) = (1+e^{-i\xi})^{m+1}\, C'' \prod_{j=1}^{n} (e^{-i\xi} - w_j)(e^{-i\xi} - \bar{w}_j).$$

Comment: This problem was, admittedly, not easy. We have in fact proved a classical result of Riesz, see page 172 in Daubechies's book.

(3) Clearly, $M(\xi) = |m_0(\xi)|^2 > 0$ for $|\xi| \le \frac{\pi}{2}$, which means that we have orthonormality. The vanishing of moments is ensured by the factor $(1+e^{-i\xi})^{m+1}$.