TMA 4170 Exercises Week 10 2018

Solution

B&N 5.8. We set $V_j := \{ f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset [-2^j\pi, 2^j\pi] \}$ (which we recognize as a Paley-Wiener space).

(a) Property 1 is obvious. By Plancherel and the fact that compactly supported functions form a dense subset of $L^2(\mathbb{R})$, 2 is also clear. If $f \in \bigcap V_j$, then $\operatorname{supp} \hat{f} \subset [-2^j\pi, 2^j\pi]$ for all $j \in \mathbb{Z}$, which is only possible if $f \equiv 0$.
We set $f_j(x) = f(2^{-j}x)$; then $\hat{f_j}(\xi) = 2^j \hat{f}(2^j\xi)$, and hence

$f \in V_j \iff f_j \in V_0$.

(b) The fact that 5 holds with $\varphi(x) := \operatorname{sinc} x$ is an immediate consequence of the sampling theorem; see also our note about the Paley-Wiener theorem.

(c) We recall that

$$\hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}}\, \chi_{[-\pi,\pi]}(\xi)$$

Hence, by Plancherel,

$$2\int \varphi(x)\,\overline{\varphi(2x-k)}\,dx$$

$$= \int \hat{\varphi}(\xi)\,\overline{\hat{\varphi}(\xi/2)}\,e^{ik\xi/2}\,d\xi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\xi/2}\,d\xi = \begin{cases} 1, & k = 0 \\ 0, & k = 2m,\ m \neq 0 \\ \left[\dfrac{e^{ik\xi/2}}{k\pi}\right]_{-\pi}^{\pi}, & k = 2m+1. \end{cases}$$

Hence we obtain

$$\varphi(x) = \varphi(2x) + \sum_{k\in\mathbb{Z}} \frac{2(-1)^k}{(2k+1)\pi}\,\varphi(2x-2k-1).$$

(d) We have

$$\psi(x) = \sum_{k\in\mathbb{Z}} (-1)^k\,\overline{p_{1-k}}\,\varphi(2x-k).$$

Now

$$p_{1-k} = \begin{cases} 1, & k = 1 \\ 0, & k = 2m+1,\ m \neq 0. \\ \dfrac{2(-1)^m}{(2m+1)\pi}, & k = -2m. \end{cases}$$

Hence

$$\psi(x) = \varphi(2x-1) + \sum_{k\in\mathbb{Z}} \frac{2(-1)^k}{(-2k+1)\pi}\,\varphi(2x-2k).$$

(e) We get

$$\ell_k = \frac{1}{2}\,\overline{p}_{-k} = \begin{cases} \frac{1}{2}, & k = 0 \\ 0, & k = 2m,\ m \neq 0 \\ \dfrac{(-1)^{m+1}}{(2m+1)}, & k = 2m+1 . \end{cases}$$

$$h_k = \frac{1}{2}(-1)^k\, p_{k+1} = \begin{cases} \frac{1}{2}, & k = -1 \\ 0, & k = 2m+1,\ m \neq -1 \\ \dfrac{(-1)^{m}}{(2m+1)}, & k = 2m . \end{cases}$$

(f) $$\tilde{\ell}_k = p_k = \begin{cases} 1, & k = 0 \\ 0, & k = 2m,\ m \neq 0 \\ \dfrac{2(-1)^{m}}{(2m+1)\pi}, & k = 2m+1 \end{cases}$$

$$\tilde{h}_k = \overline{p}_{1-k}\,(-1)^k = \begin{cases} -1, & k = 1 \\ 0, & k = 2m+1,\ m \neq 0 \\ \dfrac{2(-1)}{(2m+1)\pi}, & k = -2m . \end{cases}$$

BsN 5.17. We have

$$\varphi_m(x) = \sum p_k\, \varphi_{m-1}(2x-k),$$

with $\varphi_0(x) = \chi_{[0,1]}(x)$ and

$p_k = 0$, $|k| \leq K$, say, with $K \geq 1$.
We prove by induction on n that
$\varphi_n(x) = 0$ for $|x| > K$.
This is plainly true for $n = 0$.
Now take $|x| > K$ and assume the statement is true for $n = m$. Then if $|k| \leq K$, we have
$|2x - k| > 2K - K = K$, so
$\varphi_{m+1}(x) = 0$.

<u>B & N 5.18</u>. If $p_k = 0$ for $k > N$ and $k < 0$, then we may prove by induction on n that
$\varphi_n(x) = 0$ for $x < 0$ and $x > 0$.
Again, it is plainly true for $n = 0$.
If $x < 0$, then $2x - k < 0$ for $k \geq 0$ and so $\varphi_n(x) = 0$ for $x < 0$ implies $\varphi_{n+1}(x) = 0$ for $x < 0$.
If $x > N$, then $2x - k > 2N - N = N$ for $k \leq N$, and so $\varphi_n(x) = 0$ for $x > N$ implies $\varphi_{n+1}(x) = 0$ for $x > N$.