TMA4170 Exercises Week 3 2018

Solution

B&N 0.6 We consider

$$f_n(t) := \begin{cases} 1, & 0 \le t \le 1/n \\ 0, & \text{otherwise.} \end{cases}$$

Then $\| f_n - 0 \|_2^2 = \int_0^{1/n} 1^2 \, dt = \frac{1}{n} \to 0$

when $n \to \infty$. On the other hand, however, $\max_{0 \le t \le 1} | f_n(t) - 0 | = 1$ for every $n > 0$, whence the convergence is not uniform.

B&N 0.7 We consider

$$f_n(t) := \begin{cases} \sqrt{n}, & 0 \le t \le 1/n^2 \\ 0, & \text{otherwise.} \end{cases}$$

Then $\| f_n - 0 \|_2^2 = \int_0^{1/n^2} n \, dt = \frac{1}{n} \to 0$

when $n \to 0$. But since $f_n(0) = \sqrt{n}$, we have that in fact $f_n(0) \to \infty$.

B&N 0.8. Consider for example

$$f_n(t) := \begin{cases} \frac{1}{\sqrt{n}}, & 0 \le t \le n \\ 0, & \text{otherwise.} \end{cases}$$

Then f_n converges uniformly to 0, but $f_n \not\to 0$ in L^2 since $\| f_n \|_2 = 1$.

B&N 0.11. By integration by parts

$$\int_0^\pi f(t)\cos t\,dt = [f(t)\sin t]_0^\pi$$

$$- \int_0^\pi f'(t)\sin t\,dt.$$ Here the first term on the right-hand side 0 since $\sin 0 = \sin \pi = 0$. Hence the result follows.

B&N 0.15. We set

$$\phi(x) := \begin{cases} 1, & 0 \le 1 < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\psi(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2} \\ -1, & 1/2 \le x < 1 \\ 0, & \text{otherwise}. \end{cases}$$

We see by inspection that $\phi(x)$, $\psi(x)$, $\psi(2x)$, $\psi(2x-1)$ constitute an orthonormal system. Hence the orthogonal projection Pf of $f(x) := x$ onto this system in $L^2[0,1]$ is:

$$Pf(x) = \frac{1}{2}\phi(x) - \frac{1}{4}\psi(x) - \frac{1}{16}\psi(2x)$$

$$- \frac{1}{16}\psi(2x-1).$$

B & N 0.23. Let $e_1, \dots, e_n$ be a set of n orthonormal vectors in an inner product space. Suppose now that $a_1 e_1 + \dots + a_n e_n = 0$ for some scalars $a_1, \dots, a_n$. Since $\langle a_1 e_1 + \dots + a_n e_n, e_k \rangle = a_k$ for $k = 1, \dots, n$, we infer from this that $a_1 = \dots = a_n = 0$, hence the vectors are linearly independent.