THE PALEY–WIENER THEOREM AND FOURIER ANALYSIS

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The purpose of this note is to elaborate on the relation between complex analysis and Fourier analysis. We will do this by giving a complex analytic proof of a classical theorem of Paley and Wiener. As a matter of fact, we will see that this proof gives an alternate approach to the Plancherel identity and the L^2 theory of Fourier series and transforms.

We say that an entire function f is of exponential type if there exists a positive constant c such that $|f(z)| \leq \exp(c|z|)$ when $|z|$ is large enough. The type of such an f is the number

$$\limsup_{|z|\to\infty} \frac{\log|f(z)|}{|z|}.$$

We notice that for example $\cos az$ is of exponential type a when $a > 0$, all polynomials are of 0 type, while e^{-z^2} grows too fast to be of exponential type.

The Paley–Wiener theorem reads as follows.

Theorem 1. *If f is of exponential type at most Ω and*

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty,$$

then there exists a function $\widehat{f}$ in $L^2(-\Omega,\Omega)$ such that

$$f(z) = \int_{-\Omega}^{\Omega} \widehat{f}(\xi) e^{i\xi z} d\xi.$$

Proof. We may assume without loss of generality that $\Omega = \pi$. Introduce an auxiliary function

$$f_0(z) := \int_{-1/2}^{1/2} f(z+t)\,dt.$$

We claim that if $f_0 \equiv 0$, then $f \equiv 0$. To see this, consider

$$f_0(z+\varepsilon) - f_0(z) = \int_{1/2}^{1/2+\varepsilon} (f(z+t) - f(z+t-1))\,dt$$

for $0 < \varepsilon < 1/2$ and assume this is identically zero. We let $\varepsilon \to 0$ and use the continuity of f to conclude that we must have

$$f(z+1/2) = f(z-1/2),$$

which means that f is 1-periodic. But this is incompatible with the assumption that f is in $L^2(\mathbb{R})$ unless $f \equiv 0$.

We infer from the preceding argument that it suffices to show that

$$f_0(z) = \int_{-\pi}^{\pi} \widehat{f_0}(\xi) e^{i\xi z} d\xi \tag{1}$$

since then

$$f_0(z) = \int_{-1/2}^{1/2} g(z+t)\,dt$$

with

$$g(z) = \int_{-\pi}^{\pi} \frac{\xi/2}{\sin(\xi/2)} \widehat{f_0}(\xi) e^{i\xi z} d\xi,$$

and consequently

$$\widehat{f}(\xi) = \frac{\xi/2}{\sin(\xi/2)} \widehat{f_0}(\xi).$$

To prove (1), we begin by noting that

(i) f_0 is also of exponential type at most π (obvious).

(ii) $|f_0(x)|^2 \le \int_{-\infty}^{\infty} |f(t)|^2 dt := M$ for every real point x (by Cauchy–Schwarz).

(iii) $\sum_{n=-\infty}^{\infty} |f_0(n)|^2 \le M$ (again by Cauchy–Schwarz).

(iv) $|f_0(x+iy)| \le Me^{\pi|y|}$.

Here (iv) follows from (i) and (ii) by a version of the maximum modulus principle known as the Phragmén–Lindelöf principle (see below).

We will now prove that

$$f_0(z) = \sum_{n=-\infty}^{\infty} f_0(n) \frac{\sin \pi(z-n)}{\pi(z-n)}, \tag{2}$$

with uniform convergence on compact subsets of $\mathbb{C}$. Indeed, it is clear that the series on the right-hand side of (2) converges uniformly on compact sets and hence represents an entire function of exponential type π. Denote this function by g, i.e.,

$$g(z) := \sum_{n=-\infty}^{\infty} f_0(n) \frac{\sin \pi(z-n)}{\pi(z-n)}.$$

Now (iv) implies that $(f_0(z) - g(z))/\sin \pi z$ is a bounded entire function, whence a constant by Liouville's theorem. But since both $f_0(x) \to 0$ and $g(x) \to 0$ when $x \to \pm\infty$, this constant must be 0, and so we have established (2).

Finally, we observe that (2) can be written

$$\begin{aligned} f_0(z) &= \sum_{n=-\infty}^{\infty} f_0(n) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi(z-n)} d\xi \\ &= \int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} f_0(n) e^{-in\xi} \right) e^{i\pi\xi z} d\xi, \end{aligned}$$

where the second equality is justified by continuity of the inner product. By (iii) and orthogonality of the complex exponentials $e^{in\xi}$, the desired function $\widehat{f_0}$ has thus been identified. □

We now explain how (i) and (ii) imply (iv). Consider the function $F_\varepsilon(z) := f_0(z)e^{i(\pi+\varepsilon)z}$ for $\varepsilon > 0$. Then the assumption that f_0 is of exponential type $\le \pi$ implies that $|F_\varepsilon(iy)| \le C_\varepsilon$ for $y \ge 0$ and some positive constant C_ε. Now consider the function

$$G_\delta(z) := F_\varepsilon(z) e^{-\delta(e^{-i\pi/4} z)^{3/2}}.$$

This function is analytic in the first quadrant and bounded and continuous in the closed quadrant. We also have that $G_\delta(z) \to 0$ uniformly on the circular arc $|z| = R$, $0 \le \arg z \le \pi/2$, when $R \to \infty$. Hence the maximum modulus principle implies that $|G_\delta(z)| \le \max(M, C_\varepsilon)$. Letting $\delta \to 0$, we see that $|F_\varepsilon(z)| \le \max(M, C_\varepsilon)$ in the first quadrant. Applying the same argument in the second quadrant, we conclude that this inequality holds in the upper half-plane. Now suppose that $|F_\varepsilon(iy_0)| > M$ for some point $y_0 > 0$. Considering $F_\varepsilon(z)(z+i)^{-\delta}$ and letting $\delta \to 0$, we see that this violates the maximum modulus principle. We conclude that, for every $\varepsilon > 0$, $|F_\varepsilon(z)| \le M$ in the upper half-plane. Hence $|f_0(x+iy)| \le Me^{\pi y}$ for $y > 0$. Applying the same argument to $f_0(-z)$, we obtain (iv).

The above argument is a variant of the maximum modulus principle for unbounded domains, known as the Phragmén–Lindelöf principle. You may find it as a theorem in textbooks on complex analysis. The general principle is that a function bounded on the rays of a sector is bounded throughout the sector whenever the growth is sufficiently moderate compared with the angular opening of the sector. The Phragmén–Lindelöf theorem makes the relation between the growth rate and the angular opening precise.

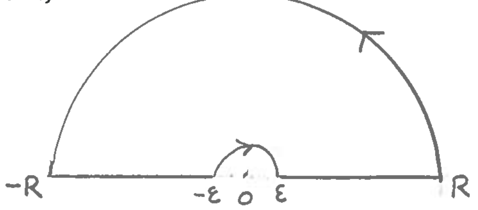
A final comment: The usual proofs of the Paley–Wiener theorem rely on the Plancherel identity in one way or another, see for instance [1, Sect. 3.3]. The point I wished to make by writing this note, is that it is logically more satisfactory to do it in reverse order: Prove the Paley–Wiener without assuming any result from Fourier analysis (except orthogonality of the complex exponentials), and obtain as a fringe benefit the classical L^2 theory!

Consequences

1. The Paley–Wiener theorem implies that (i)–(iv) hold when f_0 is replaced by f. It follows that also (2) holds with f_0 replaced by f. This we know as the sampling theorem, but notice that we have now proved it based only on complex analysis. You may check that the convergence is uniform on $\mathbb{R}$.
2. We may argue that $\widehat{f}$ is unique in different ways. You may for instance repeat the argument involving the Fejér kernel and approximate identities.
3. Point 2 shows immediately that the exponentials $e^{in\xi}/\sqrt{2\pi}$ constitute an orthonormal basis for $L^2(-\pi,\pi)$.
4. Some computations that you find in the exercises below, show that the functions
$$\frac{\sin\pi(x-n)}{\pi(x-n)}$$
constitute an orthonormal sequence in $L^2(\mathbb{R})$ and that
$$\int_{-\infty}^{\infty}\frac{\sin\pi x}{\pi x}e^{-i\xi x}\,dx = \chi_{[-\pi,\pi]}(\xi).$$
By point 3, we then get Plancherel's identity for functions with Fourier transforms supported on $[-\pi,\pi]$. If we now let $\Omega\to\infty$, we get it for a general L^2 function. We also get the L^2 inversion formula for the Fourier transform.

1. Exercises

(1) Show that $\int_{-\infty}^{\infty}\frac{\sin\pi x}{\pi x}dx = 1$. (Hint: Apply Cauchy's theorem to the function $e^{i\pi z}/(\pi z)$ in a domain as shown in the figure below.)



(2) Show that $\int_{-\infty}^{\infty}\frac{\sin^2\pi x}{\pi^2x^2}dx = 1$. (Hint: Integration by parts and problem 1.)

(3) Show that

$$\int_{-\infty}^{\infty} \frac{\sin\pi(x-m)}{\pi(x-m)} \frac{\sin\pi(x-n)}{\pi(x-n)} dx = 0$$

when $m \neq n$. (Hint: Make a partial fractions decomposition of the denominator.)

(4) Show that

$$\int_{-\infty}^{\infty} \frac{\sin\pi x}{\pi x} e^{-i\xi x} dx = \begin{cases} 1, & |\xi| < \pi \\ 1/2, & |\xi| = \pi \\ 0, & |\xi| > 0. \end{cases}$$

(Hint: Use trigonometric identities and problem 1.)

References

[1] H. Dym and H. P. McKean, *Fourier Series and Integrals*, Academic Press, New York–London, 1972.