TMA4170 FOURIER ANALYSIS, WEEK 5, 2018: THE SCHWARTZ CLASS AND THE POISSON SUMMATION FORMULA

We define the Schwartz class $\mathscr{S}$ as the set of functions f in $C^\infty(\mathbb{R})$ such that $(|x|+1)^m f^{(n)}(x) \to 0$ when $|x| \to \infty$ for all nonnegative integers m, n.

(1) Verify that the Fourier transformation $\mathscr{F}$ maps $\mathscr{S}$ into and onto itself.

(2) Consider the function

$$f(x) := \begin{cases} 0, & x \le 0 \\ \exp\left(-x^{-1}e^{1/(x-1)}\right), & 0 < x < 1 \\ 1, & x \ge 1. \end{cases}$$

Verify that functions of the form $f(\lambda(x-a)) - f(\lambda(x-b))$ are in $\mathscr{S}$ when $\lambda > 0$ and $a < b$ are real points. How will this function look like when λ is large?

(3) For what we do next, it will be convenient to use the following definition of the Fourier transform:

$$\widehat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi} dx.$$

Suppose that f is in $\mathscr{S}$. Show that

$$\sum_{n=-\infty}^{\infty} f(x+n)$$

is a 1-periodic C^∞ function, and use this to establish the identity

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{2\pi i k x}.$$

(4) We may reformulate the result of (3) as

$$\sum_{n=-\infty}^{\infty} f(x+nT) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \widehat{f}(k/T) e^{2\pi i k x/T}$$

for every $T > 0$. Now show that you obtain the inversion formula

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i x\xi} d\xi$$

for f in $\mathscr{S}$ by letting $T \to \infty$ in the preceding formula.

(5) By setting $x = 0$ in (3) we get the famous Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \widehat{f}(k).$$

Use it to verify Jacobi's identity

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} = t^{-1/2} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/t}, \quad t > 0.$$