TMA4170 FOURIER ANALYSIS, WEEK 4, 2018: THE DIRICHLET KERNEL AND THE FEJÉR KERNEL

Our aim is to understand the difference between the Dirichlet kernel

$$D_n(x) := \frac{1}{2\pi}\frac{\sin(n+1/2)x}{\sin(x/2)}$$

and the Fejér kernel

$$F_n(x) := \frac{1}{n}\sum_{k=0}^{n-1} D_n(x),$$

as well as how $F_n(x)$ is used to approximate functions by trigonometric polynomials. We will use the notation

$$(f*g)(x) := \int_{-\pi}^{\pi} f(y)g(x-y)\,dy,$$

where both f and g are assumed to be 2π-periodic.

(1) Verify that $\int_{-\pi}^{\pi} D_n(x)\,dx = \int_{-\pi}^{\pi} F_n(x) = 1$.

(2) Show that

$$F_n(x) = \frac{1}{2\pi}\frac{\sin^2(nx/2)}{n\sin^2(x/2)},$$

and conclude that $F_n(x) \geq 0$ for all x. (Hint: Use a suitable finite geometric series.)

(3) Show that there exists a positive constant c such that

$$\int_{-\pi}^{\pi} |D_n(x)|\,dx \geq c\log n$$

for every $n \geq 1$. (Hint: Make use of the fact that $|\sin(x/2)| \leq |x|/2$.)

(4) Show that there exists a constant $C > 0$ such that

$$\int_{\varepsilon\leq|x|\leq\pi} F_n(x)\,dx \leq \frac{C}{\max(1,\varepsilon n)},$$

and hence this integral tends to 0 when $n \to \infty$ for every fixed ε.

(5) Show that for every f in $C(\mathbb{T})$ we have

$$\|f - F_n * f\|_\infty \to 0.$$

(Hint: Write

$$f(x) = \int_{-\pi}^{\pi} f(x)F_n(x-y)\,dy$$

and use the properties of F_n deduced above, as well as the fact that $f(x) - f(x-\delta)$ tends uniformly to 0 when $\delta \to 0$ (uniform continuity of continuous functions on closed intervals).)

(6) We set $f_\delta(x) := f(x-\delta)$. Take for granted that $C(\mathbb{T})$ is a dense subspace of $L^1(\mathbb{T})$. Use this to show that $\|f - f_\delta\|_1 \to 0$ when $\delta \to 0$ for every f in $L^1(\mathbb{T})$.

(7) Argue similarly as in point (5) to show that

$$\|f - f * F_n\|_1 \to 0$$

when $n \to 0$ for every f in $L^1(\mathbb{T})$.

Remarks

- The properties established above show that F_n is a delta sequence (or an approximate identity); every sequence of functions K_n with the same properties ($\int K_n = 1$, $K_n \geq 0$, $\int_{\varepsilon \leq |x| \leq \pi} K_n \to 0$) will lead to the same approximation results as those established in points (5) and (7) (with F_n replaced by K_n). Another important example of such a kernel is the Poisson kernel which you find in Exercise 40 on page 90 in B & N. You may check for yourself that the argument used in point (5) above gives that the expression in (1.44) in B & N tends uniformly to $f(\phi)$ when $r \nearrow 1$ if f is continuous.
- You may appreciate the simplicity of the results obtained in points (5) and (7) by taking into account two famous results reflecting how delicate and intricate the general question of pointwise convergence of Fourier series is: Kolmogorov's example (1926) of an L^1 function whose Fourier series diverges at every point of the circle and Carleson's theorem (1966) of almost everywhere convergence of Fourier series of functions in L^2.
- You conclude from point (5) that the set of trigonometric polynomials constitutes a dense subspace of $C(\mathbb{T})$. This fact allows you to prove the famous Weierstrass approximation theorem, which asserts that every continuous function on a closed interval can be approximated arbitrarily well (in the uniform sense) by an ordinary polynomial. (Here, to get from trigonometric polynomials to ordinary polynomials, you just use Taylor's theorem to approximate the complex exponential function.)