TMA4170 FOURIER ANALYSIS, WEEK 16, 2018: MISCELLANEOUS APPLICATIONS AND RESULTS

(1) We know that $e^{-\pi x^2}$ is its own Fourier transform (here we assume '2π' to be placed in the exponent of the complex exponential). Can you find other functions with the property that they are a constant multiple of themselves? Explain why these constant multiples can be only ± 1 and $\pm i$.

(2) We define the Fejér kernel F_R of the real line to be the function

$$F_R(x) := R\left(\frac{\sin \pi R x}{\pi R x}\right)^2$$

for $R > 0$. Compute the Fourier transform of F_R, and show that the functions F_R constitute a family of 'good kernels' when $R \to \infty$.

(3) Use the Poisson summation formula to prove that

$$\sum_{n=-\infty}^{\infty} F_N(x+n) = \frac{1}{N}\left(\frac{\sin \pi N x}{\pi x}\right)^2,$$

which is the Fejér kernel on the circle.

(4) Recall the definition of the gamma function,

$$\Gamma(s) := \int_0^{\infty} x^{s-1} e^{-x}\, dx.$$

Establish the functional equation $\Gamma(s+1) = s\Gamma(s)$ and compute $\Gamma(1/2)$ and $\Gamma(3/2)$.