TMA4170 FOURIER ANALYSIS, WEEK 15, 2018: MORE APPLICATIONS OF FOURIER METHODS

(1) We have seen that the isoperimetric inequality can be rephrased analytically in terms of the following statement:

$$\int_0^1 x(t)y'(t)\,dt \le \frac{1}{4\pi}\left(\int_0^1 \sqrt{x'(t)^2+y'(t)^2}\,dt\right)^2,$$

where x and y are 1-periodic smooth and real-valued functions. Check that by a change of variables we may write this as

$$\int_0^{2\pi} x(t)y'(t)\,dt \le \frac{1}{4\pi}\left(\int_0^{2\pi} \sqrt{x'(t)^2+y'(t)^2}\,dt\right)^2 \tag{1}$$

for smooth 2π-periodic functions x and y, and verify that we may assume without loss of generality that $\int_0^{2\pi} x(t)\,dt = 0$. Given a smooth 2π-periodic real-valued function f with mean 0, set $x = f$ and $y(t) = \int_0^t f(\tau)\,d\tau$, and deduce Wirtinger's inequality

$$\int_0^{2\pi} |f(t)|^2\,dt \le \int_0^{2\pi} |f'(t)|^2\,dt \tag{2}$$

from (1).

(2) Assume a given closed curve Γ with length L is parametrized by $x = x(t)$, $y = y(t)$, $0 \le t \le 2\pi$, such that

$$|x'(t)|^2 + |y'(t)|^2 \equiv L^2/(2\pi)^2$$

for $0 \le t \le 2\pi$. Now deduce the isoperimetric inequality from (2). (Hence the isoperimetric inequality and Wirtinger's inequality in the form (2) are equivalent.)

(3) Deduce the following special case of Euler's summation formula:

$$\sum_{n=1}^N f(n) = \int_0^N f(x)\,dx + \int_0^N (x-[x])f'(x)\,dx,$$

where $[x]$ denotes the integer part of x and f is in $C^1([0,N])$. Hint: Integrate by parts on $[n-1,n]$ and sum up.

(4) Prove that for $a \ne 0$ and $0 < \sigma < 1$, the sequence $\langle an^\sigma\rangle$ is equidistributed in $[0,1)$. Hint: By Weyl's criterion, you should estimate the sum

$$\sum_{n=1}^N e^{2\pi i k a n^\sigma}.$$

To this end, use the result of the preceding problem to arrive at the integral

$$\int_0^N e^{2\pi i k a x^\sigma}\,dx,$$

and estimate this by a suitable integration by parts.