TMA4170 FOURIER ANALYSIS, WEEK 12, 2018: SOME APPLICATIONS OF FOURIER METHODS

(1) The point of this problem is to finish the computation of expected number of returns to 0 for a random walk. Take for granted that a sphere of radius r in $\mathbb{R}^d$ has surface area

$$\frac{2(\pi)^{d/2} r^{d-1}}{\Gamma(d/2)}.$$

Use this to show that the function

$$f_d(x) := \frac{\cos 2\pi x_1 + \cdots + \cos 2\pi x_d}{d}$$

satisfies

$$\int_{\mathbb{T}^d} \frac{dx}{1 - f_d(x)} \begin{cases} = \infty, & d = 2 \\ < \infty, & d \geq 3. \end{cases}$$

(2) The purpose of this problem is to verify the last part of the proof of the Central Limit Theorem. Let f_n be a sequence of positive functions such that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x) k(x)\, dx = \int_{-\infty}^{\infty} g(x) k(x)\, dx$$

for every function k in the Schwartz class, where g is a positive and bounded function. Show that then

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b g(x) k(x)\, dx$$

whenever $a < b$. (Hint: Approximate $\chi_{[a,b]}(x)$ from above and below by nonnegative Schwartz functions.)

(3) This problem gives a combinatorial application of Parseval's identity. We wish to show that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

To verify this, use the binomial formula to write the function

$$(1 + e^{2\pi i x})^n$$

as a trigonometrical polynomial. Then check that

$$|1 + e^{2\pi i x}|^{2n} = (e^{\pi i x} + e^{-\pi i x})^{2n},$$

and conclude by using Parseval's identity and applying the binomial formula to the latter expression.