NTNU – Trondheim
Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4170 Fourier Analysis**

Academic contact during examination: Ole Fredrik Brevig

Phone: 979 63 419

Examination date: May 23, 2018

Examination time (from–to): 09:00–13:00

Permitted examination support material: One yellow A4-sized sheet of paper stamped by the Department of Mathematical Sciences. On this sheet the student may write whatever he/she wants. Allowed calculators: Casio fx-82ES PLUS and Casio fx-82EX, Citizen SR-270X and Citizen SR-270X College, and Hewlett Packard HP30S.

Other information:
This examination paper consists of 9 problems that will all be given the same weight.

Language: English

Number of pages: 2

Number of pages enclosed: 0

Checked by:

Date Signature

Problem 1

a) Show that

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(2\pi nx)}{\pi n} = x$$

for $-1/2 < x < 1/2$. (You should in particular explain why we know that the series converges on this interval.)

b) Use the result in part a) to show that

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} = \frac{\pi}{4} \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Problem 2 Use an induction argument to show that e^{-x^4}, $x \in \mathbb{R}$, belongs to the Schwartz class.

Problem 3 Recall Weyl's theorem which says that a sequence $x_1, x_2, \ldots$ of real numbers is equidistributed mod 1 if and only if we have, for all integers $k \neq 1$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k x_n} = 0.$$

Use this theorem to show that the sequence θn, $n = 1, 2, \ldots$, is equidistributed mod 1 if and only if θ is an irrational number.

Problem 4

a) Let ϕ be the scaling function of a multiresolution analysis. Deduce the scaling relation

$$\phi(x) = \sum_{k=-\infty}^{\infty} p_k \phi(2x-k),$$

and express the coefficients p_k as an inner product involving the scaling function ϕ.

b) Show that if ϕ is compactly supported, then p_k is nonzero only for finitely many k.

c) Set

$$m_0(\xi) := \frac{1}{2} \sum_{k=-\infty}^{\infty} p_k e^{ik\xi},$$

take the Fourier transform of the scaling relation, and deduce formally the formula

$$\widehat{\phi}(\xi) = \widehat{\phi}(0) \prod_{j=1}^{\infty} m_0\left(-2^{-j}\xi\right).$$

(Here we use the convention that $\widehat{\phi}(\xi) := \int_{-\infty}^{\infty} \phi(x) e^{-ix\xi} dx$.)

Problem 5

a) The Poisson kernel P_r on the circle $\mathbb{T}$ is defined as

$$P_r(t) := \frac{1 - r^2}{1 - 2r\cos t + r^2}$$

for $0 \leq r < 1$. Show that we may write

$$P_r(t) = \mathrm{Re}\left(\frac{2}{1 - re^{it}} - 1\right),$$

and use this to prove that

$$\frac{1}{2\pi} \int_0^{2\pi} P_r(t) dt = 1$$

whenever $0 \leq r < 1$.

b) Let f be a continuous 2π-periodic function and set

$$f_r(t) := \frac{1}{2\pi} \int_0^{2\pi} f(x) P_r(t - x) dx.$$

Show that $f_r(t) \to f(t)$ uniformly on $[0, 2\pi]$ when $r \to 1^-$. (Hint: Recall that a continuous function on a closed interval is in fact uniformly continuous.)