Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4170 Fourier Analysis
Spring 2017

Exercise set Exam solution

1 **a)** Starting from $\hat{f}(\xi)$, we find its Fourier inverse transform

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} d\xi \\ &= \frac{1}{2} \int_{-\infty}^{\infty} e^{-|\xi|+ix\xi} d\xi \\ &= \frac{1}{2} \left(\int_{-\infty}^{0} e^{(1+ix)\xi} d\xi + \int_{0}^{\infty} e^{(-1+ix)\xi} d\xi \right) \\ &= \frac{1}{2} \left(\frac{1}{1+ix} + \frac{1}{1-ix} \right) = \frac{1}{1+x^2}. \end{aligned}$$

b) Using Plancherel's theorem, we find that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2} dx &= \int_{-\infty}^{\infty} \mathscr{F}\left(e^{-x^2}\right)(\xi) \mathscr{F}\left(\frac{1}{1+x^2}\right)(\xi) d\xi \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}} e^{-\xi^2/4} \sqrt{\frac{\pi}{2}} e^{-|\xi|} d\xi \\ &= \sqrt{\pi} \int_{0}^{\infty} e^{-(\xi^2+4\xi)/4} d\xi \\ &= \sqrt{\pi} e \int_{0}^{\infty} e^{-(\xi+2)^2/4} d\xi \\ &= 2\sqrt{\pi} e \int_{1}^{\infty} e^{-x^2} dx, \end{aligned}$$

with a change of variables (and slight abuse of notation) $x = (\xi+2)/2$. From this, we can see that $C = 2\sqrt{\pi}e$.

2 A pretty straightforward calculation gives us:

$$\begin{aligned} \mathscr{F}[x * y]_k &= \sum_{j=0}^{n-1} (x * y)_j \overline{w}^{jk} = \sum_{j=0}^{n-1} \sum_{l=0}^{n-1} x_l y_{j-l} \overline{w}^{jk} = \sum_{l=0}^{n-1} \sum_{j=0}^{n-1} x_l \overline{w}^{lk} y_{j-l} \overline{w}^{(j-l)k} \\ &= \sum_{l=0}^{n-1} x_l \overline{w}^{lk} \sum_{j=0}^{n-1} y_{j-l} \overline{w}^{(j-l)k} = \sum_{l=0}^{n-1} x_l \overline{w}^{lk} \sum_{m=0}^{n-1} y_m \overline{w}^{mk} = \mathscr{F}[x]_k \mathscr{F}[y]_k \end{aligned}$$

3 Using the definitions of Fourier transform and derivatives for distributions, we find

$$\begin{aligned}\widehat{\delta^{(k)}}(\phi) &= \delta^{(k)}(\hat{\phi}) = (-1)^k\delta(\hat{\phi}^{(k)}) = (-1)^k\hat{\phi}^{(k)}(0)\\ &= (-1)^k\frac{\mathrm{d}^k}{\mathrm{d}\xi^k}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x)\mathrm{e}^{-i\xi x}\mathrm{d}x\Big|_{\xi=0}\\ &= \int_{-\infty}^{\infty}\frac{(ix)^k}{\sqrt{2\pi}}\phi(x)\mathrm{d}x.\end{aligned}$$

So, in the sense of distributions, $\widehat{\delta^{(k)}} = (ix)^k/\sqrt{2\pi}$.

4 **a)** We can compute $\phi(x)$ using the inverse Fourier transform:

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\phi}(\xi)\mathrm{e}^{i\xi x}\mathrm{d}\xi = \frac{1}{2\pi}\int_{a-\pi}^{a+\pi}\mathrm{e}^{i\xi x}\mathrm{d}\xi = \frac{1}{i2\pi x}(\mathrm{e}^{i(a+\pi)x} - \mathrm{e}^{i(a-\pi)x}) = \mathrm{e}^{iax}\frac{\sin(\pi x)}{\pi x}.$$

b) It is easier to work with the Fourier transform in this case, so we find

$$\int_{-\infty}^{\infty}\phi(x)\mathrm{d}x = \sqrt{2\pi}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x)\mathrm{e}^{i0x}\mathrm{d}x = \sqrt{2\pi}\hat{\phi}(0) = 1.$$

c) Using the equivalent condition in Theorem 5.18 in B&N, we see that

$$2\pi\sum_{k\in\mathbb{Z}}|\hat{\phi}(\xi+2\pi k)|^2 = \sum_{k\in\mathbb{Z}}|\chi_K(\xi+2\pi k)|^2 = 1,$$

since K is an interval of width 2π, so only one of the indices gives an argument for which χ_K is nonzero.

d) Once again, we turn to the Fourier criteria since the scaling function may be hard to work with. By theorem 5.19 in B&N, the scaling condition is equivalent to

$$\hat{\phi}(\xi) = \hat{\phi}\left(\frac{\xi}{2}\right)P\left(\mathrm{e}^{-i\xi/2}\right),$$

where

$$P(z) = \frac{1}{2}\sum_{k\in\mathbb{Z}}p_k z^k.$$

In our case, this means

$$\chi_K(\xi) = \chi_K\left(\frac{\xi}{2}\right)P\left(\mathrm{e}^{-i\xi/2}\right).$$

Since $\chi_K(\xi/2)$ is nonzero on an interval containing K, this is equivalent to

$$\chi_K(\xi) = P\left(\mathrm{e}^{-\frac{i\xi}{2}}\right) = \frac{1}{2}\sum_{k\in\mathbb{Z}}p_k\mathrm{e}^{-ik\xi/2},$$

or

$$2\chi_K(-2\xi) = \sum_{k\in\mathbb{Z}}p_k\mathrm{e}^{ik\xi}.$$

With $K = [a-\pi, a+\pi]$, one can see that $-2\xi \in K$ iff $\xi \in [-1/2(a+\pi), -1/2(a-\pi)]$, and so the above is again equivalent to

$$2\chi_{[-\frac{1}{2}(a+\pi),-\frac{1}{2}(a-\pi)]}(\xi) = \sum_{k\in\mathbb{Z}} p_k e^{ik\xi}.$$

Using the supplied Fourier series components, it is evident that

$$p_0 = 1, \quad p_k = \frac{2}{\pi k} e^{-ia/2} \sin\left(\frac{\pi k}{2}\right), k = \pm 1, \pm 2, \pm 3, \ldots$$

Alternatively, one may write, with $m \in \mathbb{Z}$:

$$p_k = \begin{cases} 1, & k = 0 \\ \dfrac{2(-1)^m}{\pi(2m+1)} e^{-ia/2}, & k = 2m+1 \\ 0, & k = 2m. \end{cases}$$

e) The wavelet is defined as

$$\psi(x) = \sum_{k\in\mathbb{Z}} (-1)^k \overline{p_{1-k}} \phi(2x-k),$$

and using the scaling constants from the previous exercise, we get

$$\overline{p_{1-k}} = \begin{cases} 1, & k = 1 \\ \dfrac{2(-1)^n}{\pi(2n+1)} e^{ia/2}, & k = -2n \\ 0, & k = 2n+1. \end{cases}$$

From this, we can see that

$$\psi(x) = -\phi(2x-1) + e^{ia/2} \sum_{n\in\mathbb{Z}} \frac{2(-1)^n}{\pi(2n+1)} \phi(2x+2n).$$

We can continue and find a closed form for ψ. First, let us change the summation variable to $m = -n-1$. The expression for the wavelet becomes

$$\psi(x) = -\phi(2x-1) + e^{ia/2} \sum_{m\in\mathbb{Z}} \frac{2(-1)^m}{\pi(2m+1)} \phi(2x-2m-2).$$

Next, consider the scaling relation for ϕ, which with the p_k inserted becomes

$$\phi(x) = \phi(2x) + e^{-ia/2} \sum_{m\in\mathbb{Z}} \frac{2(-1)^m}{\pi(2m+1)} \phi(2x-2m-1).$$

Then, compare the two expressions:

$$e^{-ia/2}\psi(x) = -e^{-ia/2}\phi(2x-1) + \sum_{m\in\mathbb{Z}} \frac{2(-1)^m}{\pi(2m+1)} \phi(2x-2m-2)$$
$$e^{ia/2}\phi(x-1/2) = e^{ia/2}\phi(2x-1) + \sum_{m\in\mathbb{Z}} \frac{2(-1)^m}{\pi(2m+1)} \phi(2x-2m-2)$$

and observe that

$$e^{-ia/2}\psi(x) + e^{-ia/2}\phi(2x-1) = e^{ia/2}\phi(x-1/2) - e^{ia/2}\phi(2x-1).$$

Solving this for $\psi(x)$ gives us

$$\begin{aligned}
\psi(x) &= e^{ia}\phi(x-1/2) - (e^{ia}+1)\phi(2x-1) \\
&= e^{ia}e^{ia(x-1/2)}\frac{\sin(\pi x - \pi/2)}{\pi x - \pi/2} - (e^{ia}+1)e^{ia(2x-1)}\frac{\sin(2\pi x - \pi)}{2\pi x - \pi} \\
&= \frac{(1+e^{-ia})e^{i2ax}\sin(2\pi x) - 2e^{ia/2}e^{iax}\cos(\pi x)}{2\pi x - \pi}
\end{aligned}$$