NTNU
Norwegian University of
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Department of Mathematical Sciences

Examination paper for **TMA4170 Fourier Analysis**

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Examination time (from–to): 09:00–13:00

Permitted examination support material: One yellow A4-sized sheet of paper stamped by the Department of Mathematical Sciences. On this sheet the student may write whatever he/she wants. Allowed calculators: Casio fx-82ES PLUS and Casio fx-82EX, Citizen SR-270X and Citizen SR-270X College, and Hewlett Packard HP30S.

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Informasjon om trykking av eksamensoppgave
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Problem 1

The Fourier transform and its inverse are given by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-i\xi x}\mathrm{d}x, \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi)\mathrm{e}^{i\xi x}\mathrm{d}\xi.$$

a) Show that the inverse Fourier transform of

$$\hat{f}(\xi) = \sqrt{\frac{\pi}{2}}\mathrm{e}^{-|\xi|}$$

is

$$f(x) = \frac{1}{1+x^2}.$$

b) Show that

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-x^2}}{1+x^2}\mathrm{d}x = C\int_{1}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x$$

and determine the constant C. You may use without proof that

$$\mathcal{F}\left[\mathrm{e}^{-ax^2}\right](\xi) = \frac{1}{\sqrt{2a}}\mathrm{e}^{-\xi^2/(4a)}, \quad a > 0.$$

Problem 2

Let x and y be n-periodic sequences. Define their convolution for $k = 0, ..., n-1$ by

$$(x * y)_k = \sum_{l=0}^{n-1} x_l y_{k-l},$$

and the discrete Fourier transform by

$$\mathcal{F}[x]_k = \sum_{j=0}^{n-1} x_j \overline{w}^{jk}, \qquad w = \mathrm{e}^{2\pi i/n}.$$

Show that

$$\mathcal{F}[x * y]_k = \mathcal{F}[x]_k \mathcal{F}[y]_k.$$

Problem 3

Show that

$$\widehat{\delta^{(k)}} = \frac{(ix)^k}{\sqrt{2\pi}}$$

in the sense of distributions, where $\delta^{(k)}$ denotes the kth derivative of Dirac's delta distribution. Recall that the Fourier transform $\widehat{T}$ of a distribution T is given by

$$\widehat{T}(\phi) = T(\hat{\phi}).$$

Problem 4

The Fourier transform and its inverse are given by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-i\xi x} \mathrm{d}x, \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) \mathrm{e}^{i\xi x} \mathrm{d}\xi.$$

Let $K = [a - \pi, a + \pi]$ with $-\pi < a < \pi$, and define

$$\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \chi_K(\xi) = \begin{cases} \frac{1}{\sqrt{2\pi}}, & \xi \in K \\ 0, & \xi \notin K. \end{cases}$$

a) Compute $\phi(x)$.

b) Show that

$$\int_{-\infty}^{\infty} \phi(x) \mathrm{d}x = 1.$$

c) Show that ϕ satisfies the orthogonality condition

$$\int_{-\infty}^{\infty} \phi(x - k) \overline{\phi(x - l)} \mathrm{d}x = \delta_{kl}, \quad k, l \in \mathbb{Z}.$$

d) Show that ϕ satisfies the scaling condition

$$\phi(x) = \sum_{k \in \mathbb{Z}} p_k \phi(2x - k)$$

by determining the scaling constants p_k.

Hint: You may use that the function $2\chi_{[-\frac{1}{2}(a+\pi),-\frac{1}{2}(a-\pi)]}(\xi)$ has (complex) Fourier series components

$$c_0 = 1, \quad c_k = \frac{2}{\pi k} \mathrm{e}^{-ia/2} \sin\left(\frac{\pi k}{2}\right), \quad k = \pm 1, \pm 2, \pm 3, \ldots$$

e) Assuming that ϕ defines a multiresolution analysis, determine the corresponding wavelet ψ and show that

$$\psi(x) = \mathrm{e}^{ia}\phi\left(x - 1/2\right) - (\mathrm{e}^{ia} + 1)\phi(2x - 1).$$