FOURIER ANALYSIS 16. V. 2015

① $e^{ix} + e^{3ix} + e^{5ix} + \dots + e^{2015ix}$

$$= e^{ix}\left[1 + e^{2ix} + e^{4ix} + \dots + e^{2\cdot 1007 ix}\right] \quad (\text{geom. series!})$$

$$= e^{ix}\,\frac{1-(e^{2xi})^{1008}}{1-e^{2xi}} = \frac{1-\cos(2016x) - i\sin(2016x)}{\underbrace{e^{-ix}-e^{+ix}}_{=-2i\sin(x)}}$$

Take the real part of both sides:

$$\begin{cases} S(x) = \cos(x) + \cos(3x) + \dots + \cos(2015x) \\ \quad = \dfrac{1}{2}\,\dfrac{\sin(2016x)}{\sin(x)}, \quad x \neq N\pi \\ S(N\pi) = (-1)^N \cdot 1008 \end{cases}$$

Thus $S(x) = 0 \iff x = \dfrac{n\pi}{2016}$ & $2016 \nmid n$

The values $x = 0, \pm 2016\pi, \pm 2\cdot 2016\pi, \dots$ are excluded.

④ Start with $\phi_0 = \phi_{HAAR} = \chi_{(0,1)}$. Then calculate recursively

$$\phi_j(x) = \sum_{k=0}^{3} p_k\, \phi_{j-1}(x), \quad j = 1, 2, 3, \dots$$

It is known that $\phi = \lim_{j\to\infty} \phi_j$. (Another method is based on infinite products.) See textbook

(4) First, calculate directly

$$\widehat{e^{-a|x|}} = \sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+\omega^2}, \quad \widehat{e^{-b|x|}} = \sqrt{\frac{2}{\pi}}\,\frac{b}{b^2+\omega^2}$$

By Plancherel's formula

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{a}{(a^2+y^2)}\frac{b}{(b^2+y^2)}\,dy = \int_{-\infty}^{\infty}\widehat{e^{-a|x|}}\;\widehat{e^{-b|x|}}\,d\omega$$

$$= \int_{-\infty}^{\infty} e^{-a|x|}e^{-b|x|}\,dx = \int_{-\infty}^{\infty} e^{-(a+b)|x|}\,dx = 2\int_{0}^{\infty} e^{-(a+b)x}\,dx$$

$$= \frac{2}{a+b}. \quad \text{Thus} \int_{-\infty}^{\infty}\frac{dy}{(a^2+y^2)(b^2+y^2)} = \underline{\frac{\pi}{ab(a+b)}}$$

where $a>0$, $b>0$.

(5) We know that $\hat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}}\chi_{(-\pi,\pi)}(\omega)$

(It is easy to calculate the Fourier transform of a characteristic function!) Now

$$\widehat{\psi(x-k)} = e^{-i\omega k}\,\hat{\psi}(\omega)$$

Thus

$$\langle \psi_j, \psi_k\rangle \overset{\text{Plancherel}}{=} \langle \hat{\psi}_j, \hat{\psi}_k\rangle = \int_{-\infty}^{\infty} e^{-i\omega j}e^{+i\omega k}\,|\hat{\psi}(\omega)|^2\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega(k-j)}\cdot 1^2\,d\omega = \begin{cases} 1, & k=j \\ 0, & k\neq j. \end{cases}$$

③ $\hat{T}(\phi) = T(\hat{\phi})$ by definition

$$T(\hat{\phi}) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\hat{\phi}(\omega) - \hat{\phi}(-\omega)}{\omega}\, d\omega$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-i\omega x} - e^{+i\omega x}}{\omega\sqrt{2\pi}}\, \phi(x)\, dx\, d\omega$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \phi(x)\left(\lim_{c\to\infty}\int_{-c}^{c} \frac{e^{-i\omega x} - e^{i\omega x}}{\omega\sqrt{2\pi}}\, d\omega\right) dx$$

Remark: The precaution with $c \to \infty$ is not needed.

$$= \frac{-2i}{2\sqrt{2\pi}} \int \phi(x) \underbrace{\lim_{c\to\infty}\int_{-c}^{c} \frac{\sin(\omega x)}{\omega}\, d\omega}_{\pi\, \text{sign}(x)}$$

$$= -i\pi \int_{-\infty}^{\infty} \text{sign}(x)\, \phi(x)\, dx \cdot \frac{1}{\sqrt{2\pi}}$$

In symbols $\hat{T} = -i\pi/\sqrt{2\pi}\ \text{sign}(x\omega)$

$$= -i\sqrt{\pi/2}\ \text{sign}(x\omega)$$

$$\int_{-\infty}^{\infty} \frac{\sin(cx)}{x}\, dx = \begin{cases} \pi, & c > 0 \\ 0, & c = 0 \\ -\pi, & c < 0 \end{cases}$$

(6) First we must kill the square:

$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}.$$

Then

$$\int_a^b f(x)\cos^2(nx+n^3)\,dx = \frac{1}{2}\int_a^b f(x)\,dx + \frac{1}{2}\int_a^b f(x)\cos(2nx+2n^2)\,dx$$

<u>Claim:</u> $\displaystyle\lim_{n\to\infty}\int_a^b f(x)\cos(2nx+2n^2)\,dx = 0.$

<u>Proof:</u>

$$\left|\int_a^b f(x)e^{\pm i(2nx+2n^2)}\,dx\right| = \left|e^{\pm i\cdot 2n^2}\int_a^b f(x)e^{\pm 2inx}\,dx\right| = 1\left|\int_a^b f(x)e^{\pm 2inx}\,dx\right|$$

$\longrightarrow 1\cdot|0| = 0$ by the Riemann–Lebesgue lemma.

The claim follows from Euler's formula. □

<u>Answer:</u>

$$\lim_{n\to\infty}\int_a^b f(x)\cos^2(nx+n^3)\,dx = \frac{1}{2}\int_a^b f(x)\,dx$$