Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4170 Fourieranalyse

Exam, December 11, 2004, Time: 9:00–13:00

Contact during exam: Helge Holden, phone 92038625

Grades will be announced January 7, 2005

Aids: One A4-sized sheet of paper stamped by the Department of Mathematical Sciences. On this sheet the student can write whatever he or she wants. No other aids.

Problem 1

(a) Let $f(x) = \arctan(x)$. Does f define a regular distribution?

(b) Is f in the space $L^p(\mathbb{R})$ for any $1 \leq p \leq \infty$?

(c) Does f define a tempered distribution?

(d) Compute $\mathcal{F}(f') = \widehat{f'}$, the Fourier transform of the derivative of f. You may use that

$$\int \frac{e^{ipx}}{1+x^2} dx = \pi e^{-|p|}.$$

(e) Show that the distribution $\hat{f}$ (or more formally $\mathcal{F}(T_f)$) satisfies

$$\xi \hat{f} = \frac{1}{2i} e^{-2\pi|\xi|}.$$

(f) You may use the following facts without proof:
The distribution $\text{pv}(1/x)$ satisfies equation

$$\xi \cdot \text{pv}(1/\xi) = 1.$$

Furthermore, the distributional equation

$$\xi S = g, \quad S \in \mathcal{D}'$$

where $g \in L^1_{\text{loc}}(\mathbb{R})$ with $g(\xi)/\xi \in L^1_{\text{loc}}(\mathbb{R})$ has as general solution

$$S = \frac{g(\xi)}{\xi} + \alpha\delta,$$

where α is an arbitrary complex number, and δ denotes Dirac's delta distribution.
Use this to show that

$$\hat{f} = \frac{1}{2i\xi}\left(e^{-2\pi|\xi|} - 1\right) + \frac{1}{2i}\,\text{pv}(1/\xi) + \alpha\delta$$

for some complex number α.

(g) Show that $\alpha = 0$.

Problem 2

(a) Consider the analog filter $g = A(f)$ where

$$g'' + 3g' + 2g = f$$

on the set $\mathcal{S}$ of Schwartz functions. Determine the impulse response h such that we can write the filter as $g = h * f$.

(b) Is the filter realizable (causal) and stable?

Problem 3

(a) Consider the analog filter $g = A(f)$ where

$$g = f'$$

on the set $\mathcal{S}'$ of tempered distributions. Determine the impulse response h such that we can write the filter as $g = h * f$.

(b) Is the filter realizable (causal)?

Some useful formulas

$$
\begin{aligned}
\hat{f}(t) &= \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n t/a}, \quad f(a+t) = f(t),\\
c_n &= \frac{1}{a}\int_0^a f(t) e^{-2\pi i n t/a}\, dt, \quad n \in \mathbb{Z},\\
\mathcal{F}(f)(\xi) &= \hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x}\, dx,\\
\mathcal{F}^{-1}(f)(x) &= \int_{\mathbb{R}} f(\xi) e^{2\pi i \xi x}\, d\xi,\\
\mathcal{F}(x^k e^{-ax} u(x)/k!) &= 1/(a + 2\pi i \xi)^{k+1}, \quad a \in \mathbb{C},\ \mathrm{Re}(a) > 0,\ k = 0, 1, 2, \dots,\\
\mathcal{F}(x^k e^{ax} u(-x)/k!) &= -1/(-a + 2\pi i \xi)^{k+1}, \quad a \in \mathbb{C},\ \mathrm{Re}(a) > 0,\ k = 0, 1, 2, \dots,\\
\mathcal{F}(e^{-ax^2}) &= \sqrt{\pi/a}\, e^{-(\pi\xi)^2/a}, \quad a \in \mathbb{R},\ a > 0,\\
\mathcal{F}(u(x)) &= \frac{1}{2}\delta + \frac{1}{2\pi i}\mathrm{pv}(1/\xi),\\
\mathcal{F}(\mathrm{sign}(x)) &= \frac{1}{\pi i}\mathrm{pv}(1/\xi).
\end{aligned}
$$