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**Exercise set 9 solution**

1 **B&N: 5.5** If $\phi$ has compact support, then by the definition of compact support, $\phi(x) = 0$ outside some bounded set. That is, there exist $L \in \mathbb{R}$ such that $\phi(x) = 0$ for $|x| > L$. Theorem 5.6 states that the scaling coefficients $p_k$ are given by

$$p_k = 2 \int_{-\infty}^{\infty} \phi(x)\overline{\phi(2x-k)}.\mathrm{d}x$$

The support of $\phi(x)$ is contained in $[-L, L]$, while the support of $\phi(2x-k)$ is contained in $\left[-\frac{L}{2}+\frac{k}{2}, \frac{L}{2}+\frac{k}{2}\right]$, and $p_k$ can be nonzero only if the two supports overlap, which is clearly not the case once $k > \frac{3L}{2}$ or $k < -\frac{3L}{2}$. Thus, only a finite amount of $p_k$ are nonzero.

2 **B&N: 5.6** We are assuming that $\{V_j\}_{j\in\mathbb{Z}}$ is an MRA, and that it has a continuous and compactly supported scaling function $\phi$. As a consequence, an orthonormal basis for $V_j$ is $\{\phi_{jk}\}$, where $\phi_{jk}(x) = 2^{\frac{j}{2}}\phi(2^j x - k)$.

**a)** The orthogonal projection $u_j$ onto the space $V_j$ of the step function $u$ is given as

$$u_j(x) = \sum_{k\in\mathbb{Z}} \langle u, \phi_{jk}\rangle \phi_{jk}(x),$$

where

$$\langle u, \phi_{jk}\rangle = \int_{-\infty}^{\infty} u(x)\phi_{jk}(x)\mathrm{d}x = 2^{\frac{j}{2}} \int_0^1 \phi(2^j x - k)\mathrm{d}x = 2^{-\frac{j}{2}} \int_{-k}^{2^j-k} \phi(y)\mathrm{d}y.$$

Here, we used the substitution $y = 2^j x - k$ in the last equality.

**b)** Since $\phi$ has compact support, there exists an $L \in \mathbb{R}$ such that $\phi(x) = 0$ when $|x| > L$. Thus,

$$\int_{-L}^{L} \phi(x)\mathrm{d}x = \int_{-\infty}^{\infty} \phi(x)\mathrm{d}x = 0.$$

Now, we turn to the projection coefficients

$$\langle u, \phi_{jk}\rangle = 2^{-\frac{j}{2}} \int_{-k}^{2^j-k} \phi(y)\mathrm{d}y.$$

Note that if $k \geq 2^j + L$, the upper limit of the integral is to the left of the support of $\phi$ and thus the integral is zero. Similary, if $k \leq -L$, the lower limit of the integral is to the right of the support of $\phi$ and the integral is zero. So the only nonzero coefficients seem to come from the terms with $-L < k < 2^j + L$. However, this range can be shortened further. Note that if $j$ is large enough (say, such that $2^j > 2L$), then the terms with $L < k < 2^j - L$ also become zero since in this case

$$\langle u, \phi_{jk} \rangle = 2^{-\frac{j}{2}} \int_{-L}^{2^j - k} \phi(y) \mathrm{d}y = 2^{-\frac{j}{2}} \int_{-L}^{L} \phi(y) \mathrm{d}y = 0.$$

So, with $j$ large enough, the coefficients can only be nonzero if $-L < k < L$ or if $2^j - L < k < 2^j + L$. That is, the number of nonzero coefficients is less than $4L$. (This is a slight difference from the solution in B&N, which states this number as $2L$.) Now, we can make a global estimate on the coefficients:

$$\left| \langle u, \phi_{jk} \rangle \right| \leq 2^{-\frac{j}{2}} \int_{-k}^{2^j - k} |\phi(y)| \mathrm{d}y \leq 2^{-\frac{j}{2}} \int_{-\infty}^{\infty} |\phi(y)| \mathrm{d}y,$$

and use this to find that

$$\| u_j \| \leq \sum_{k \in \mathbb{Z}} \left| \langle u, \phi_{jk} \rangle \right| \| \phi_{jk} \| \leq 4L 2^{-\frac{j}{2}} \int_{-\infty}^{\infty} |\phi(y)| \mathrm{d}y.$$

Here, we have used that $\|\phi_{jk}\| = 1$ by the orthonormality property, and that the number of nonzero coefficients cannot exceed $4L$. From this, we can see $\|u_j\| \to 0$ as $j \to \infty$, meaning the projection converges to the zero function. Therefore, since $\|u - u_j\| \to \|u\|$ we must have

$$\| u - u_j \| \geq \frac{1}{2} \| u \| = \frac{1}{2}$$

at some point.

**c)** Since $u$ is an $L^2$ function, but the projections $u_j$ do not converge to $u$, we cannot have that $L^2 = \overline{\cup_{j \in \mathbb{Z}} V_j}$, which is a contradiction to the assumption that $\{V_j\}_{j \in \mathbb{Z}}$ is an MRA. Thus, we must have

$$\int_{-\infty}^{\infty} \phi(x) \mathrm{d}x \neq 0.$$

3 **B&N: 5.7** Since, in general, $E$ and $O$ can be complex, we write $E = a + ib$, $O = c + id$, with $a, b, c, d \in \mathbb{R}$. The two equations now state that

$$|E|^2 + |O|^2 = a^2 + b^2 + c^2 + d^2 = 2$$
$$E + O = a + c + i(b + d) = 2.$$

From the real and imaginary parts of the second equation, we find

$$a = 2 - c$$
$$b = -d,$$

which means that

$$(2-c)^2 + d^2 + c^2 + d^2 = 2.$$

After some rearranging, one may find that this is equivalent to

$$(c-1)^2 = -d^2.$$

Since $c$ and $d$ are required to be real numbers, the only way this can work is if $c = 1$ and $d = 0$, i.e. $E = 1$ and $O = 1$.