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TMA4170 Fourier Analysis
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Exercise set 8 solution

1 B&N: 5.4

a) The set $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_0, and we can easily calculate that $\langle\phi(2x-k),\phi(2x-l)\rangle = \frac{1}{2}\delta_{kl}$. This means that

$$\begin{aligned}\delta_{l0} = \langle\phi(x-l),\phi(x)\rangle &= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} p_j\overline{p_k}\langle\phi(2x-2l-j),\phi(2x-k)\rangle \\ &= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} p_j\overline{p_k}\frac{1}{2}\delta_{j,k-2l} \\ &= \frac{1}{2}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} p_{k-2l}\overline{p_k}.\end{aligned}$$

b) This is a pretty straightforward calculation, similar to the one in a). Using equation (5.6) in B&N, we find:

$$\begin{aligned}\langle\psi_{0m},\psi_{0l}\rangle &= \frac{1}{2}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}(-1)^{j+k}\overline{p_{1-j+2m}}p_{1-k+2l}\langle\phi_{1j},\phi_{1k}\rangle \\ &= \frac{1}{2}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}(-1)^{j+k}\overline{p_{1-j+2m}}p_{1-k+2l}\delta_{jk} \\ &= \frac{1}{2}\sum_{k\in\mathbb{Z}}\overline{p_{1-k+2m}}p_{1-k+2l}.\end{aligned}$$

c) Similar to the previous point, we use (5.3) and (5.6) in B&N to find

$$\begin{aligned}\langle\phi_{0l},\psi_{0m}\rangle &= \frac{1}{2}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}(-1)^{k}p_{j-2l}p_{1-k+2m}\langle\phi_{1j},\phi_{1k}\rangle \\ &= \frac{1}{2}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}(-1)^{k}p_{j-2l}p_{1-k+2m}\delta_{jk} \\ &= \frac{1}{2}\sum_{k\in\mathbb{Z}}(-1)^{k}p_{k-2l}p_{1-k+2m}.\end{aligned}$$

2 B&N: 5.8 We consider the spaces V_j, $j\in\mathbb{Z}$, of band-limited functions f such that $\hat{f}(\lambda)=0$ outside the interval $[-2^j\pi,2^j\pi]$ with finite energy, i.e. $f\in L_2(\mathbb{R})$.

a) We show that the V_j satisfy points 1-4 in the definition of a multiresolution analysis as following:

1) Since $[-2^j\pi,2^j\pi]\subset[-2^{j+1}\pi,2^{j+1}\pi]$, we know that if $f\in V_j$, then also $f\in V_{j+1}$, meaning $V_j\subset V_{j+1}$.

2) The assumption is that $f\in V_j$ means that $f\in L_2(\mathbb{R})$ and $\hat{f}=0$ outside the interval $[-2^j\pi,2^j\pi]$. If we let $j\to\infty$, the second restriction no longer applies, and so we find $\overline{\cup_{k\in\mathbb{Z}}V_j} = L_2(\mathbb{R})$.

3) Similarly to the previous point, if we instead let $j \to -\infty$, then the support vanishes, i.e. $\cap_{k\in\mathbb{Z}} V_j$ is the space of L_2 functions supported only at the point 0. Any function $f(x)$ with value only at $x = 0$ is identifiable as the zero function in the L_2 sense, so this means that if $f \in \cap_{k\in\mathbb{Z}} V_j$, then $\hat{f} = 0$, and so $f = 0$, meaning $\cap_{k\in\mathbb{Z}} V_j = \{0\}$.
4) We will use the scaling property of the Fourier transform (theorem 2.6 in B&N), which states that if $g(t) = f(bt)$, then

$$\hat{g}(\lambda) = \frac{1}{b}\hat{f}\left(\frac{\lambda}{b}\right).$$

In particular, if $f \in V_j$, then $\hat{f}(\lambda) = 0$ for λ outside $[-2^j\pi, 2^j\pi]$, so that if $g(t) = f(2^{-j}t)$, then

$$\hat{g}(\lambda) = 2^j\hat{f}(2^j\lambda),$$

which means $\hat{g}(\lambda) = 0$ for λ outside $[-\pi,\pi]$, i.e. $f(2^{-j}t) \in V_0$. Reversing this argument, we see that if $f(2^{-j}t) \in V_0$, then $f \in V_j$.

b) First off, we can see that with $\phi(x) = \operatorname{sinc}(x)$, we have

$$\hat{\phi}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\sin(\pi x)}{\pi x}e^{-i\lambda x}\mathrm{d}x.$$

Instead of computing this integral, which requires quite a bit of computation, we shall instead consider the Fourier transform of the indicator function

$$\chi_{[a,b]}(x) = \begin{cases} 1, & x \in [a,b) \\ 0, & \text{otherwise.} \end{cases}$$

We can see that

$$\widehat{\chi_{[-\pi,\pi]}}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{-i\lambda x}\mathrm{d}x = \frac{1}{\sqrt{2\pi}\lambda}\left(\frac{e^{i\lambda\pi} - e^{-i\lambda\pi}}{i}\right) = \frac{\sqrt{2\pi}}{\pi\lambda}\sin(\pi\lambda) = \sqrt{2\pi}\phi(\lambda).$$

Applying the Fourier transform twice is the same as taking the Fourier transform and then the inverse transform with negative time, so this means

$$\sqrt{2\pi}\hat{\phi}(\lambda) = \chi_{[-\pi,\pi]}(-\lambda) = \chi_{[-\pi,\pi]}(\lambda).$$

This means that $\hat{\phi}(\lambda) = 0$ outside the interval $[-\pi,\pi]$. One can easily show that $\phi \in L_2$, and so $\phi \in V_0$. As for showing that the $\phi(x-k)$ constitute an orthonormal basis of V_0, we use the Sampling Theorem (Theorem 2.23 in B&N) with cutoff frequency $\Omega = \pi$ to see that any $f \in V_0$ can be written as

$$f(t) = \sum_{j\in\mathbb{Z}} f(j)\operatorname{sinc}(t-j) = \sum_{j\in\mathbb{Z}} f(j)\phi(t-j).$$

In addition, the translates are orthonormal by Theorem 5.18 since

$$2\pi\sum_{k\in\mathbb{Z}}|\hat{\phi}(\lambda+2\pi k)|^2 = \sum_{k\in\mathbb{Z}}\chi_{[-\pi,\pi]}(\lambda+2\pi k)^2 = 1.$$

The last equality follows since there for any λ, there is only one k for which $\lambda + 2\pi k \in [-\pi,\pi)$.

c) Using the Sampling theorem with frequency $\Omega = 2\pi$, we have

$$\phi(x) = \sum_{j\in\mathbb{Z}} \phi\left(\frac{j}{2}\right)\frac{\sin(2\pi x - j\pi)}{2\pi x - j\pi} = \sum_{j\in\mathbb{Z}} \phi\left(\frac{j}{2}\right)\phi(2x-j).$$

We can now observe that

$$\phi\left(\frac{j}{2}\right) = \frac{2}{j\pi}\sin\left(\frac{j\pi}{2}\right) = \begin{cases} 1, & j=0 \\ \frac{2}{j\pi}, & j = 1,5,9,\dots \\ -\frac{2}{j\pi}, & j = 3,7,11,\dots \\ 0, & \text{otherwise.} \end{cases}$$

So, the sum can be written over the odd indices with the zeroth term placed outside:

$$\begin{aligned} \phi(x) &= \phi(2x) + \sum_{k\in\mathbb{Z}} \phi\left(\frac{2k+1}{2}\right)\phi(2x-2k-1) \\ &= \phi(2x) + \sum_{k\in\mathbb{Z}} \frac{2(-1)^k}{(2k+1)\pi}\phi(2x-2k-1). \end{aligned} \tag{1}$$

d) From c), we can see that the scaling constants are

$$p_j = \begin{cases} 1, & j=0 \\ \frac{2(-1)^k}{(2k+1)\pi}, & j = 2k+1 \\ 0, & \text{otherwise,} \end{cases}$$

such that

$$\overline{p_{1-j}} = \begin{cases} 1, & j=1 \\ \frac{2(-1)^k}{(2k+1)\pi}, & j = -2k \\ 0, & \text{otherwise.} \end{cases}$$

This allows us to construct the wavelet ψ:

$$\begin{aligned} \psi(x) &= \sum_{j\in\mathbb{Z}} (-1)^j \overline{p_{1-j}}\phi(2x-j) \\ &= -\phi(2x-1) + \sum_{k\in\mathbb{Z}} \frac{2(-1)^k}{(2k+1)\pi}\phi(2x+2k) \end{aligned}$$

This also has a nice closed form, which can be found by replacing the summation variable k by $n = -k-1$:

$$\begin{aligned} \psi(x) &= -\phi(2x-1) + \sum_{k\in\mathbb{Z}} \frac{2(-1)^k}{(2k+1)\pi}\phi(2x+2k) \\ &= -\phi(2x-1) + \sum_{n\in\mathbb{Z}} \frac{2(-1)^n}{(2n+1)\pi}\phi(2x-2n-2). \end{aligned}$$

Now, considering that (1) gives

$$\phi\left(x-\frac{1}{2}\right) = \phi(2x-1) + \sum_{k\in\mathbb{Z}} \frac{2(-1)^k}{(2k+1)\pi}\phi(2x-2k-2),$$

we find

$$\begin{aligned}\psi(x) &= \phi\left(x-\frac{1}{2}\right) - 2\phi(2x-1) \\ &= \frac{\sin\left(\pi\left(x-\frac{1}{2}\right)\right)}{\pi\left(x-\frac{1}{2}\right)} - 2\frac{\sin(\pi(2x-1))}{\pi(2x-1)} \\ &= 2\frac{\sin(2\pi x)-\cos(\pi x)}{\pi(2x-1)}.\end{aligned}$$

e) From equations (5.15) and (5.16) in *B&N* we have the decomposition filters

$$h_j = \frac{1}{2}(-1)^j p_{j+1} = \begin{cases} -\frac{1}{2}, & j=-1 \\ \frac{(-1)^k}{(2k+1)\pi}, & j=2k \\ 0, & \text{otherwise} \end{cases}$$

and

$$l_j = \frac{1}{2}\overline{p_{-j}} = \begin{cases} \frac{1}{2}, & j=0 \\ \frac{(-1)^{k+1}}{(2k+1)\pi}, & j=2k+1 \\ 0, & \text{otherwise.} \end{cases}$$

f) From equations (5.20) and (5.21) in *B&N* we have the reconstruction filters

$$\tilde{h}_j = \overline{p_{1-j}}(-1)^j = \begin{cases} -1, & j=1 \\ \frac{(-1)^k}{(1-2k)\pi}, & j=2k \\ 0, & \text{otherwise} \end{cases}$$

and

$$\tilde{l}_j = p_k = \begin{cases} 1, & j=0 \\ \frac{2(-1)^k}{(2k+1)\pi}, & j=2k+1 \\ 0, & \text{otherwise.} \end{cases}$$

3 **B&N: 5.9**

We consider the spaces V_j, $j \in \mathbb{Z}$, of L^2 functions f such that f is continuous and piecewise linear, with corners appearing only at $k/2^j$, $k \in \mathbb{Z}$.

a) We show that the V_j satisfy points 1-4 in the definition of a multiresolution analysis as following:

1) Any $f \in V_j$ is included in V_{j+1} since it then continuous and piecewise linear, with corners appearing only on the set $\{k/2^j\}_{k\in\mathbb{Z}} \subset \{k/2^{j+1}\}_{k\in\mathbb{Z}}$, meaning $V_j \subset V_{j+1}$.

2) We wish to show that $\overline{\cup_{k\in\mathbb{Z}} V_j} = L_2(\mathbb{R})$. To this end, we approximate an $f \in L_2(\mathbb{R})$ by some $f^* \in C_0(\mathbb{R})$, the set of continuous and compactly supported functions on $\mathbb{R}$. This set is dense in $L^2(\mathbb{R})$, meaning the approximation can be made arbitrarily well, i.e. such that $\|f - f^*\| < \epsilon$. In addition, it is known that any compactly supported continuous function f^* can be approximated arbitrarily well by a piecewise linear function f^J, say $\|f^* - f^J\| < \epsilon$. So, if we take $f^J \in V_J$ for J large enough,

$$\|f - f^J\| \le \|f - f^*\| + \|f^* - f^J\| < 2\epsilon.$$

Thus, any $f \in L^2\mathbb{R}$ can be approximated arbitrarily well by a function in $\cup_{k\in\mathbb{Z}} V_j$, so $\overline{\cup_{k\in\mathbb{Z}} V_j} = L_2(\mathbb{R})$.

3) The reasoning here is similar to that of the Haar scaling function in example 5.2. Consider $f \in V_j$ as $j \to -\infty$. The spacing between corner points $k/2^j$ grows, so f is linear on increasingly large intervals. However, to have $f \in L^2(\mathbb{R})$ as $j \to -\infty$, we need f to have finite support, which is only possible if $f = 0$. Thus, $\cap_{k\in\mathbb{Z}} V_j = \{0\}$.

4) If $f \in V_j$, then $f(x)$ has corners only at the points $x = k/2^j$. By a change of variables, $f(2^{-j}x)$ has corners only at the points $x = k$, and is thus in V_0. A similar argument shows the opposite relation.

b) Note that the scaling function

$$\phi(x) = \begin{cases} x+1, & -1 \le x \le 0 \\ 1-x, & 0 < x \le 1 \\ 0, & |x| > 1 \end{cases}$$

has the property that $\phi(0) = 1, \phi(k) = 0, k \in \mathbb{Z}/\{0\}$. Thus, the translates $\phi_{0k} = \phi(x-k)$ have the property that $\phi_{0k}(j) = \delta_{jk}$. Since any $f \in V_0$ is piecewise linear with corners on the points $x = k, k \in \mathbb{Z}$, it is completely characterized by its values in the points $x = k$. So, we can write

$$f(x) = \sum_{k\in\mathbb{Z}} f(k)\phi(x-k).$$

The scaling relation follows from this fact. Considering $\phi(x)$ as a function in V_1, it is characterized by its values in the points $x = k/2$. It is zero except for the values $\phi(-1/2) = 1/2$, $\phi(0) = 1$, $\phi(1/2) = 1/2$, corresponding to $k = -1, k = 0, k = 1$. Thus,

$$\phi(x) = \frac{1}{2}\phi(2x+1) + \phi(2x) + \frac{1}{2}\phi(2x-1).$$