Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4170 Fourier Analysis
Spring 2017

Exercise set 7 solution

1 **B&N: 4.1** Expressing f in terms of $\phi(2^2x-k)$ is easy; these are the step functions of width $1/4$. As such, we can write

$$f(x) = -\phi(4x) + 4\phi(4x-1) + 2\phi(4x-2) - 3\phi(4x-3).$$

That is, $a_0^2 = -1$, $a_1^2 = 4$, $a_2^2 = 2$, $a_3^2 = -3$, and $a_k^2 = 0$ for $k < 0$, $k > 3$. Carrying out the decomposition as in Theorem 4.12, i.e. with

$$b_k^{j-1} = \frac{a_{2k}^j - a_{2k+1}^j}{2}, \qquad a_k^{j-1} = \frac{a_{2k}^j + a_{2k+1}^j}{2},$$

we find

$$b_0^1 = -\frac{5}{2}, \quad b_1^1 = \frac{5}{2}, \quad a_0^1 = \frac{3}{2}, \quad a_1^1 = -\frac{1}{2}$$

and

$$b_0^0 = 1, \quad a_0^0 = \frac{1}{2}.$$

No sketch is included; the components should be easy to draw.

2 **B&N: 4.3** We have two finite-dimensional, orthogonal subspaces A and B of an inner product space V. Both A and B are thereby inner product spaces and specifically vector spaces. The dimension of a vector space is the number of vectors in its bases. If we have that $\{v_1,\dots,v_n\}$ is a basis for A and $\{w_1,\dots,w_m\}$ is a basis for B, then $\{v_1,\dots,v_n,w_1,\dots,w_m\}$ is a basis for $A\oplus B$ since the elements are orthogonal (hence linearly independent) and thereby a minimal spanning set. Thus, we have

$$\dim A\oplus B = n+m = \dim A + \dim B.$$

However, if A and B are not orthogonal, their bases are not orthogonal and so some of the w_i may be linearly dependent on the v_i, meaning the minimal spanning set property will not be fulfilled for $\{v_1,\dots,v_n,w_1,\dots,w_m\}$ so it is not a basis. In particular, we will have

$$\dim(A+B) = \dim A + \dim B - \dim A\cap B.$$

3 **B&N: 4.5** If $f(x) = \sum_{k\in\mathbb{Z}} a_k\phi(2x-k)$ is orthogonal to all $\phi(x-l)$, then

$$\begin{aligned}0 = \langle f(x), \phi(x-l)\rangle &= \left\langle \sum_{k\in\mathbb{Z}} a_k\phi(2x-k), \phi(x-l)\right\rangle \\ &= \sum_{k\in\mathbb{Z}} a_k \int_{-\infty}^{\infty} \phi(2x-k)\phi(x-l)\mathrm{d}x \\ &= \sum_{k\in\mathbb{Z}} a_k \int_{l}^{l+1} \phi(2x-k)\mathrm{d}x.\end{aligned}$$

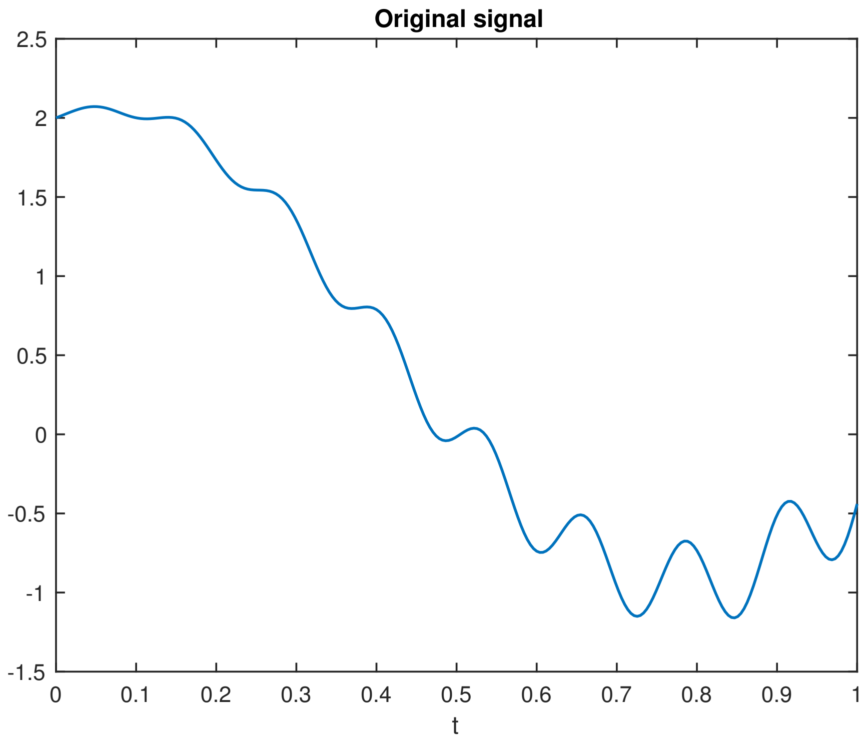
Now, we know that $\phi(2x-k) = 1$ on $[k/2, k/2+1/2]$ and $\phi(2x-k) = 0$ otherwise, so that only the integrals with $k = 2l$ and $k = 2l+1$ will be nonzero, i.e.

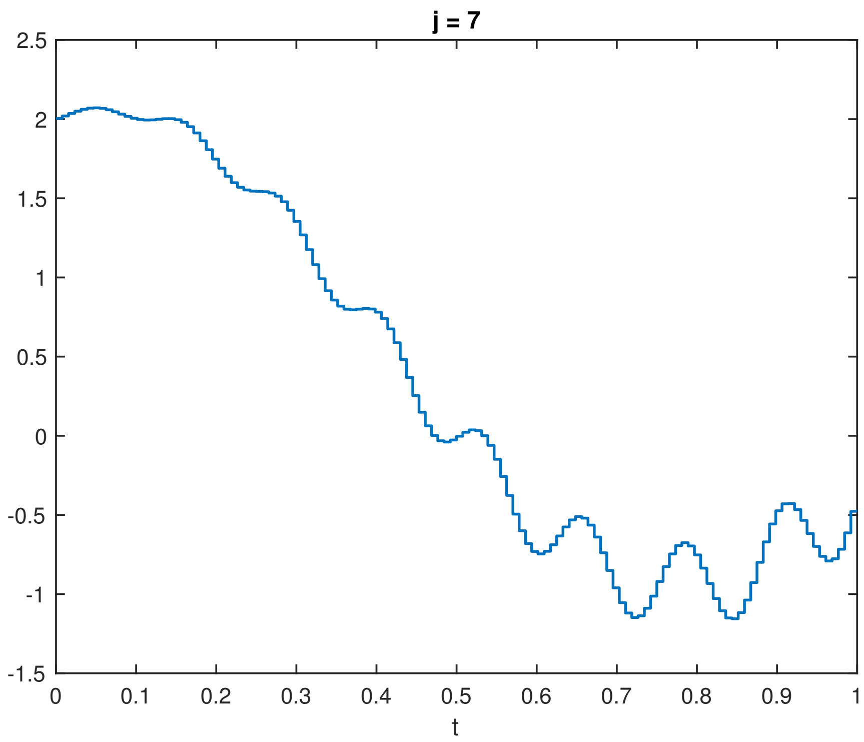
$$\begin{aligned}0 &= a_{2l}\int_{l}^{l+1} \phi(2x-2l)\mathrm{d}x + a_{2l+1}\int_{l}^{l+1} \phi(2x-2l-1)\mathrm{d}x \\ &= a_{2l}\int_{l}^{l+1/2} \mathrm{d}x + a_{2l+1}\int_{l+1/2}^{l+1} \mathrm{d}x = \frac{1}{2}(a_{2l}+a_{2l+1}).\end{aligned}$$

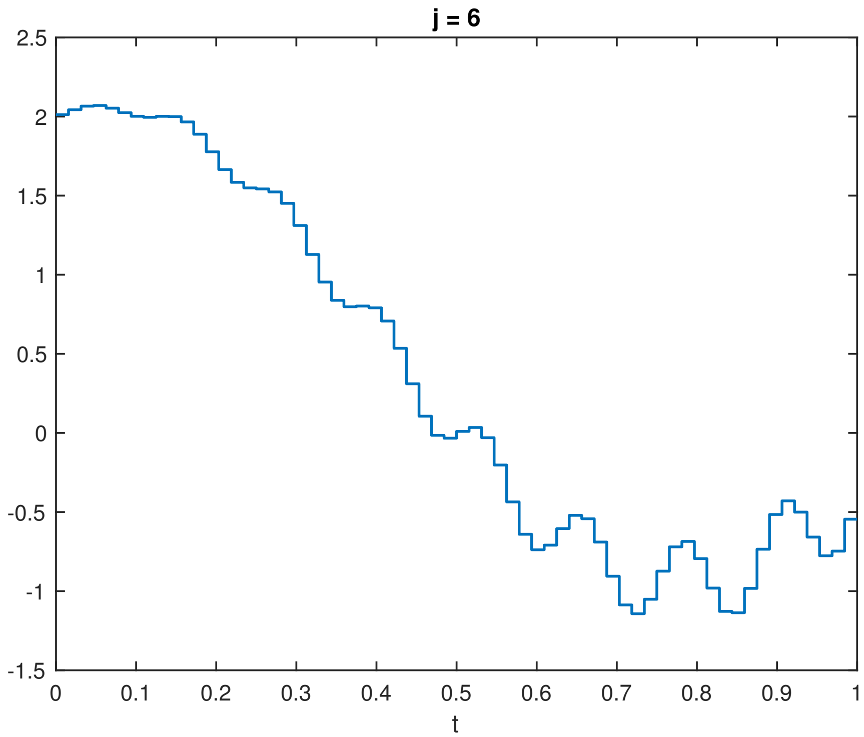
So, $a_{2l} = -a_{2l+1}$, meaning

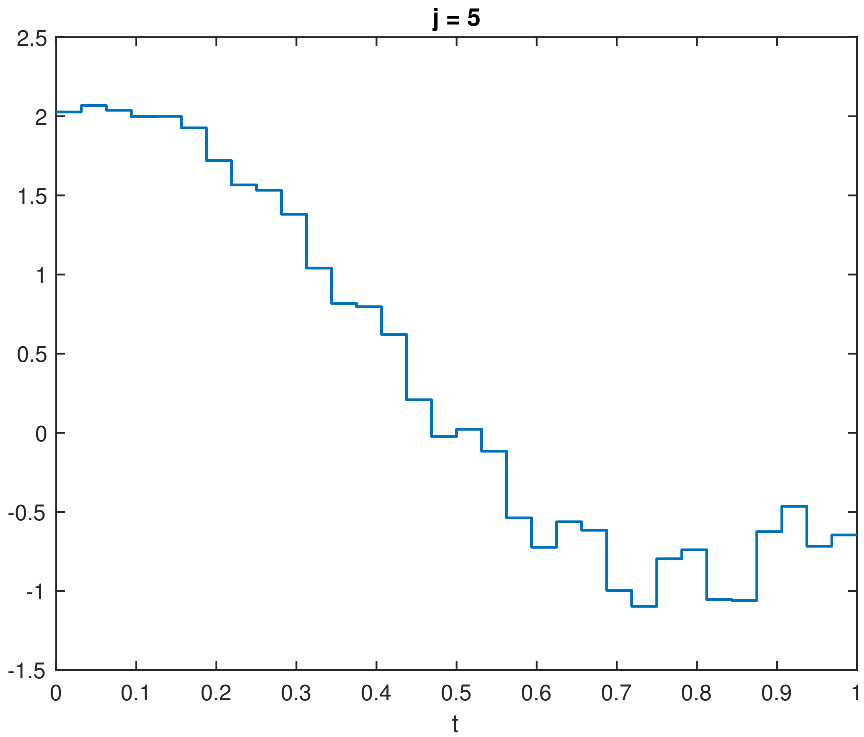
$$\begin{aligned}f(x) &= \sum_{k\in\mathbb{Z}} a_{2k}(\phi(2x-2k) - \phi(2x-2k-1)) \\ &= \sum_{k\in\mathbb{Z}} a_{2k}(\phi(2(x-k)) - \phi(2(x-k)-1)) \\ &= \sum_{k\in\mathbb{Z}} a_{2k}\psi(x-k).\end{aligned}$$

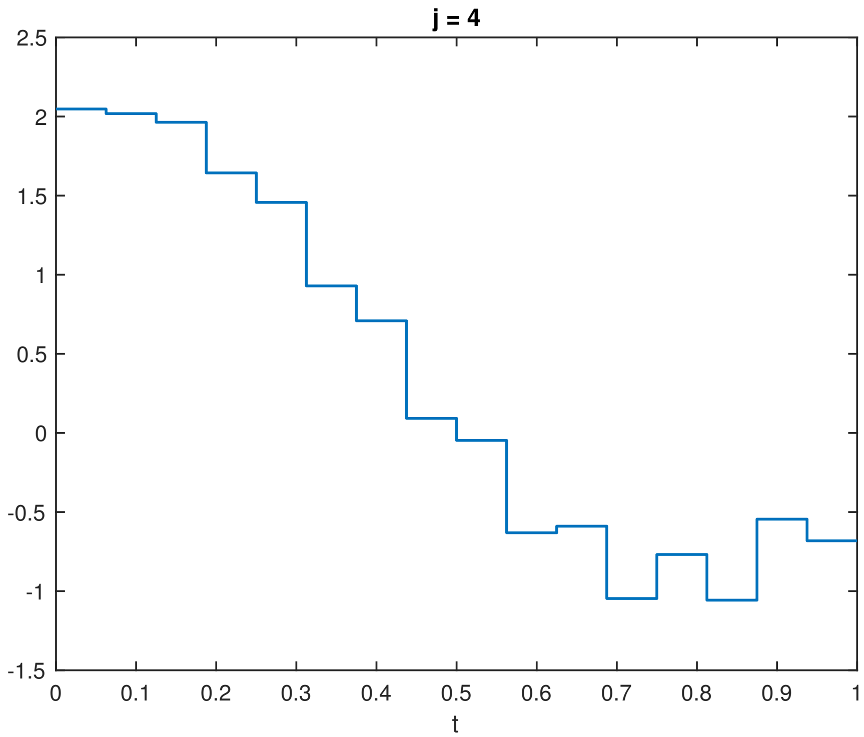
4 **B&N: 4.9** The figures below show f and its decompositions in V_j, $j = 7, \ldots, 0$.

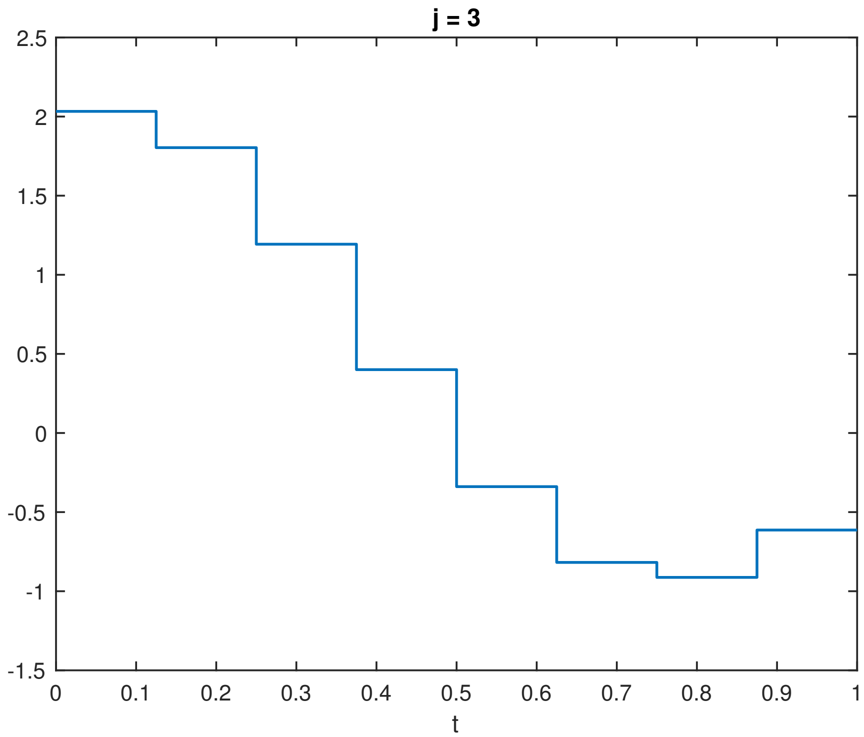

Original signal
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t

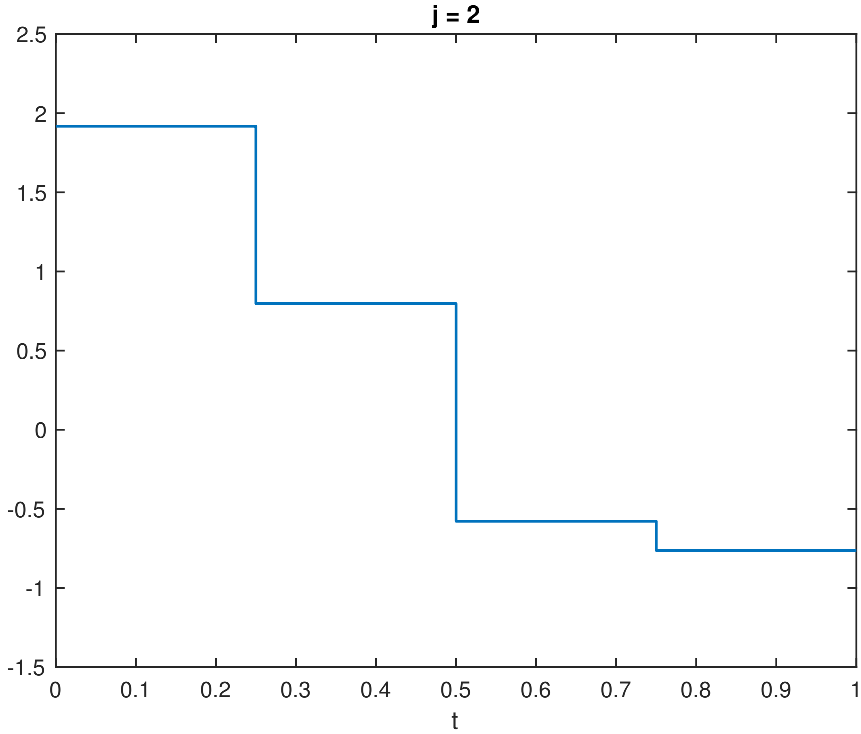

j = 7
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t

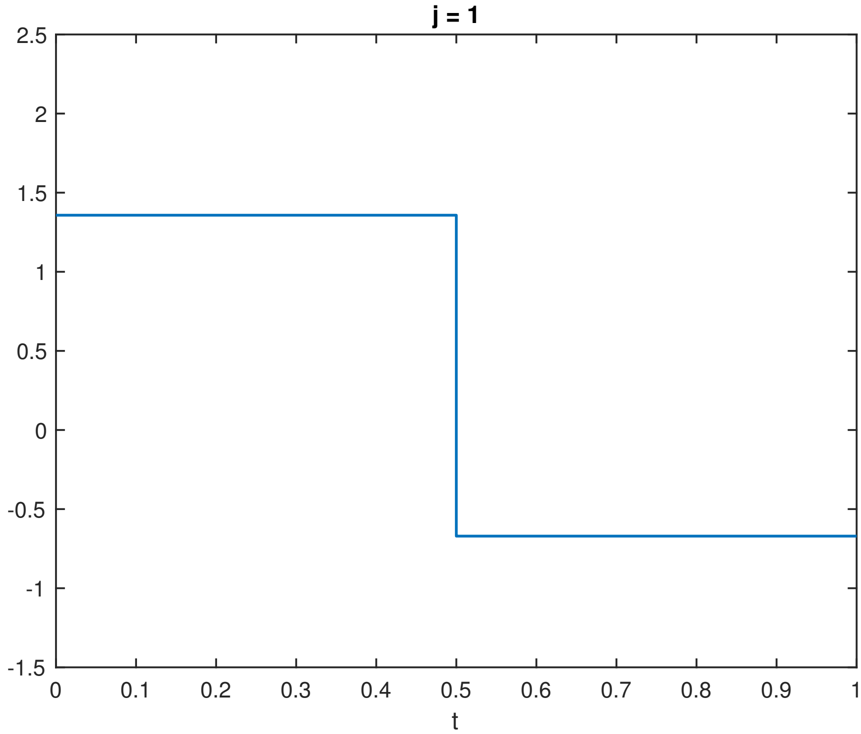

j = 6
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t

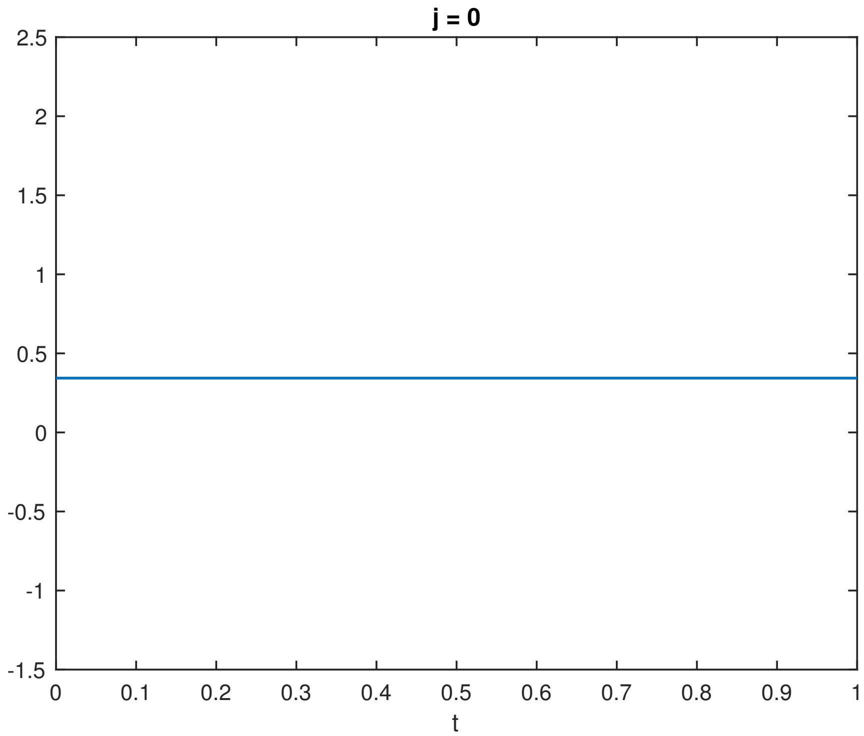

j = 5
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t


j = 4
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t


j = 3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t


j = 2
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t


j = 1
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t


j = 0
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t