Exercise set 6 solution

1 **B&N: 3.7** Due to figures, the exercise solution is found at the end of this solution proposal.

2 **B&N: 3.14**

a) If we consider n-periodic sequences u_k, the sequence $L[u]_k$ is also n-periodic as can be seen from the following:

$$L[u]_{k+n} = u_{k+n+1} - 2u_{k+n} + u_{k+n-1} = u_{k+1} - 2u_k + u_{k-1} = L[u]_k.$$

b) Since u_k is 4-periodic, it is represented by the vector $[u_1\; u_2\; u_3\; u_4]^T$. The same goes for $L[u]_k$, which is characterized by the vector $[L[u]_1\; L[u]_2\; L[u]_3\; L[u]_4]^T$. Furthermore, we can see that

$$\begin{aligned}
[L[u]_1 &= u_2 - 2u_1 + u_0 = u_2 - 2u_1 + u_4\\
[L[u]_2 &= u_3 - 2u_2 + u_1\\
[L[u]_3 &= u_4 - 2u_3 + u_2\\
[L[u]_4 &= u_5 - 2u_4 + u_3 = u_1 - 2u_4 + u_3.
\end{aligned}$$

In total, this means that

$$\begin{bmatrix} [L[u]_1 \\ [L[u]_2 \\ [L[u]_3 \\ [L[u]_4 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix},$$

meaning L is represented by the matrix, called M_4.

c) *This answer is quite long since we want to use the same procedure in part d).* We look for eigenvectors and -values of the matrix M_4. Since L is the linear transformation represented by M_4, this is the same as finding $\lambda \in \mathbb{R}$ and $u \in \mathbb{R}^4$ such that

$$L[u]_k = u_{k+1} - 2u_k + u_{k-1} = \lambda u_k, \quad k = 1,2,3,4 \tag{1}$$

where u_k can be considered a sequence in S_4. We take the DFT of this to find

$$\begin{aligned}
&e^{\frac{2\pi ik}{4}} \hat{u}_k - 2\hat{u}_k + e^{-\frac{2\pi ik}{4}} \hat{u}_k = \lambda \hat{u}_k\\
\Rightarrow &\lambda_k = 2\left(\cos\left(\frac{2\pi k}{4}\right) - 1\right), \quad k = 1,2,3,4.
\end{aligned}$$

That is, the eigenvalues are $-2, -4, -2$, and 0. Relabeling the eigenvalues as λ_j, we reinsert the now known eigenvalues into (1) to find the components u_k^j of the j'th eigenvector:

$$\begin{aligned} u^j_{k+1} - 2u^j_k + u^j_{k-1} &= \lambda_j u^j_k \\ &= 2\cos\left(\frac{2\pi j}{4}\right) u^j_k - 2u^j_k. \end{aligned}$$

That is, we need

$$u^j_{k+1} + u^j_{k-1} = 2\cos\left(\frac{2\pi j}{4}\right) u^j_k = e^{i\frac{2\pi j}{4}} u^j_k + e^{-i\frac{2\pi j}{4}} u^j_k.$$

This should be enough information for us to make the ansatz that

$$u^j_k = C_j e^{i\frac{2\pi jk}{4}} = C_j w^{jk} \quad (w = e^{i\frac{2\pi}{4}}).$$

The constants C_j are arbitrary; recall that if v is an eigenvector of a matrix A, then Cv is also an eigenvector of A for $C \in \mathbb{C}$. One can easily check (e.g. by insertion into (1)) that the eigenvectors are, indeed, correct. Thus, we have the eigenvector-eigenvalue pairs (λ_j, u^j), where

$$\lambda_j = 2\left(\cos\left(\frac{2\pi j}{4}\right) - 1\right), \quad u^j = \begin{bmatrix} u^j_1 \\ u^j_2 \\ u^j_3 \\ u^j_4 \end{bmatrix} = C_j \begin{bmatrix} w^j \\ w^{2j} \\ w^{3j} \\ w^{4j} \end{bmatrix}.$$

Now, if we choose the particular eigenvector representatives with $C_j = w^{-j}$, we have

$$u^1 = \begin{bmatrix} 1 \\ w \\ w^2 \\ w^3 \end{bmatrix} \quad u^2 = \begin{bmatrix} 1 \\ w^2 \\ w^4 \\ w^6 \end{bmatrix} \quad u^3 = \begin{bmatrix} 1 \\ w^3 \\ w^6 \\ w^9 \end{bmatrix} \quad u^4 = \begin{bmatrix} 1 \\ w^4 \\ w^8 \\ w^{12} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

So, we can see that the eigenvectors are the column vectors of the DFT matrix F_4 as seen in equation (3.2) in B&N! That is, we can rearrange the eigenvectors and eigenvalues to arrive at the eigenvalue decomposition $M = F_4 \Lambda_4 F_4^{-1}$, where

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix}, \quad \Lambda_4 = \begin{bmatrix} \lambda_4 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}.$$

This means that we can use the FFT to diagonalize M_4, as we shall now see. We write the diagonalization of M_4 in terms of multiplication with F_4 and its inverse:

$$M_4 = F_4 \Lambda_4 F_4^{-1} \Leftrightarrow F_4^{-1} M_4 = \Lambda_4 F_4^{-1} \Leftrightarrow (F_4^{-1} M_4)^H = (F_4^{-1})^H \Lambda_4 \Leftrightarrow ((F_4^{-1})^H)^{-1} (F_4^{-1} M_4)^H = \Lambda_4.$$

Here, A^H denotes the conjugate transpose of the matrix A. Since F_4 is symmetric, conjugate transposition reduces to just conjugation, i.e. $F_4^H = \overline{F_4}$. Also, as a result from basic linear algebra, conjugate transposition commutes with inversion of matrices, i.e. $(A^H)^{-1} = (A^{-1})^H$. Using these two facts simplifies the diagonalization such that we have

$$\Lambda_4 = ((F_4^{-1})^H)^{-1} (F_4^{-1} M_4)^H = ((F_4^H)^{-1})^{-1} (F_4^{-1} M_4)^H = \overline{F_4} (F_4^{-1} M_4)^H.$$

Furthermore, from the remark after Theorem 3.3, we know that $F_4^{-1} = \frac{1}{4}\overline{F_4}$, such that

$$\Lambda_4 = \frac{1}{4}\overline{F_4}(\overline{F_4}M_4)^H. \tag{2}$$

Now, consider M_4 expressed in terms of its column vectors m_1, m_2, m_3, m_4:

$$M_4 = [m_1|m_2|m_3|m_4].$$

Multiplying this by any matrix A is equivalent to multiplying its column vectors by A:

$$AM_4 = [Am_1|Am_2|Am_3|Am_4].$$

In particular, multiplying m_i with $\overline{F_4}$ is equivalent to taking the DFT of m_i. Hence, we have

$$\overline{F_4}M_4 = [\overline{F_4}m_1|\overline{F_4}m_2|\overline{F_4}m_3|\overline{F_4}m_4] = [\hat{m}_1|\hat{m}_2|\hat{m}_3|\hat{m}_4].$$

The DFT can be computed using the efficient FFT algorithm. So, in total, we can compute the diagonalization as given by (2) in three steps:
1: Compute the FFT of the columns of M_4.
2: Transpose the resulting matrix.
3: Compute the FFT of the columns of the transposed matrix and multiply by $1/4$.

Remark: This diagonalization procedure is powerful! A variant of it is used to compute finite difference approximations to the 2D Poisson equation with just $\mathcal{O}(\log n)$ operations per unknown as shown in the note at
`https://www.math.ntnu.no/emner/TMA4205/2015h/notes/Poisson2D_diag.pdf`

d) Generalizing the result from c) for all n is not too challenging; you need to follow the same steps, and find that

$$\lambda_k = 2\left(\cos\left(\frac{2\pi k}{n}\right) - 1\right), \quad k = 1, 2, ..., n.$$

Then, by the same argument as earlier, we can (by ansatz) find that the components of the eigenvectors are

$$u_k^j = C_j e^{i\frac{2\pi jk}{n}} = C_j w^{jk} \quad (w = e^{i\frac{2\pi}{n}}).$$

Normalizing the eigenvectors with $C_j = w^{-j}$ gives the eigenvalue decomposition of M_n as

$$M_n = S_n \Lambda_n S_n^{-1},$$

and the discussion follows the same track to end up with the diagonalization procedure:
1: Compute the FFT of the columns of M_n.
2: Transpose the resulting matrix.
3: Compute the FFT of the columns of the transposed matrix and multiply by $1/n$.

3 **B&N: 3.16** We want to find the Z-transform of the sequence $x = \{x_j\}_{j=-\infty}^{\infty}$ where

$$x_j = \begin{cases} \frac{1}{2^j}, & j \geq 0 \\ 0, & j < 0. \end{cases}$$

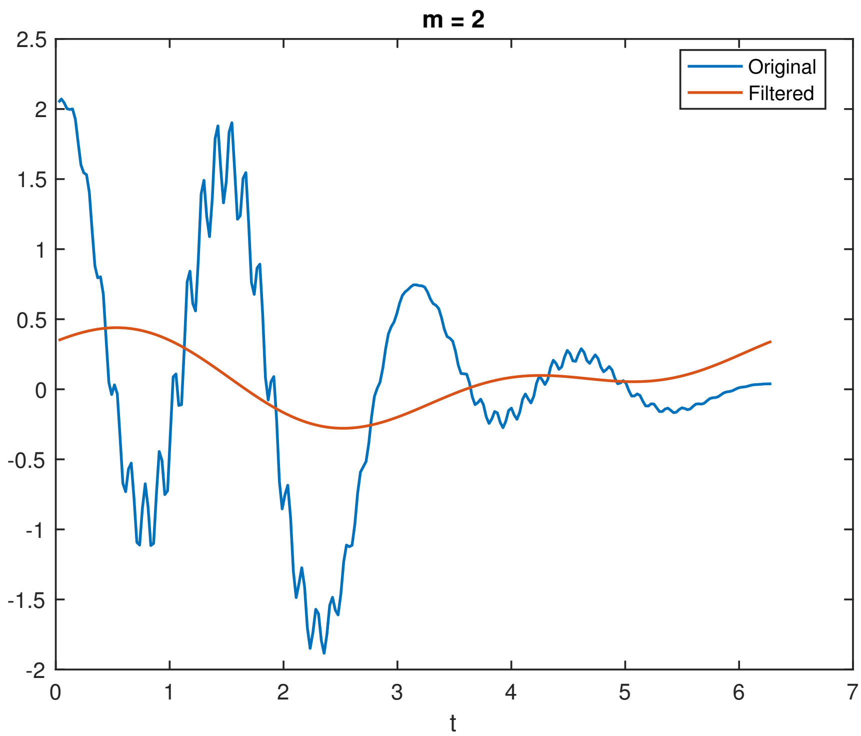
Using the definition of the Z-transform, we have

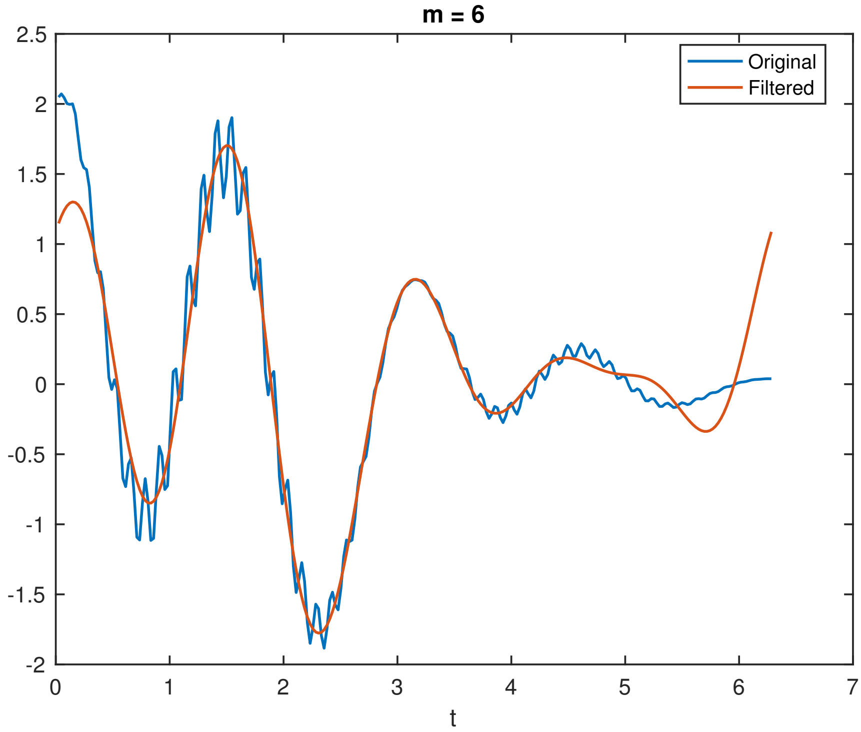
$$\hat{x}(\phi) = \sum_{j=-\infty}^{\infty} x_j e^{-ij\phi} = \sum_{j=0}^{\infty} \left(\frac{e^{-i\phi}}{2}\right)^j = \frac{1}{1-\frac{e^{-i\phi}}{2}} = \frac{1-\frac{e^{i\phi}}{2}}{\frac{5}{4}-\cos(\phi)}.$$

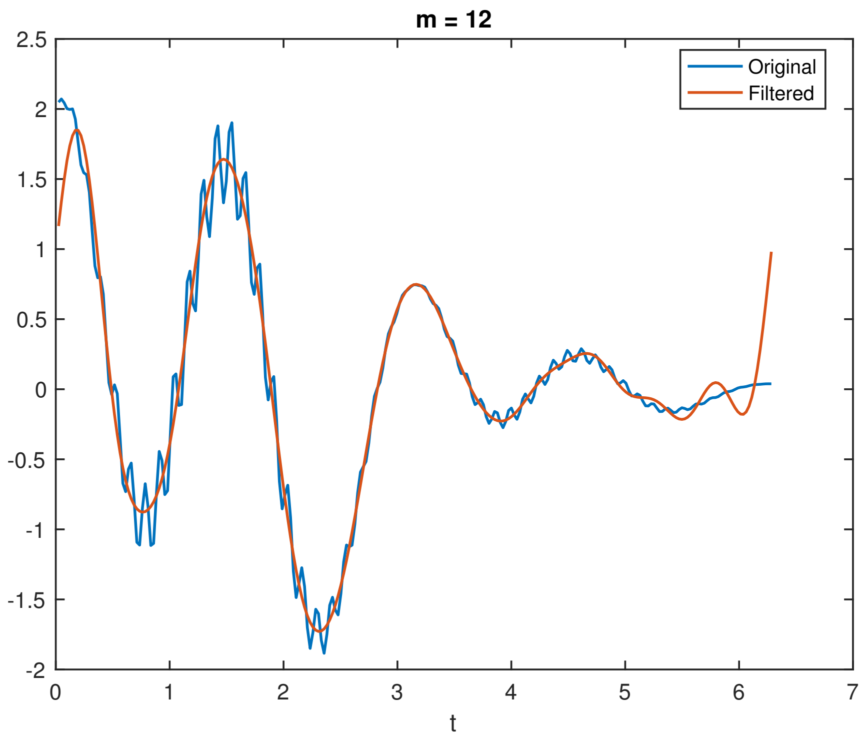
Alternatively, writing $z = e^{i\phi}$:

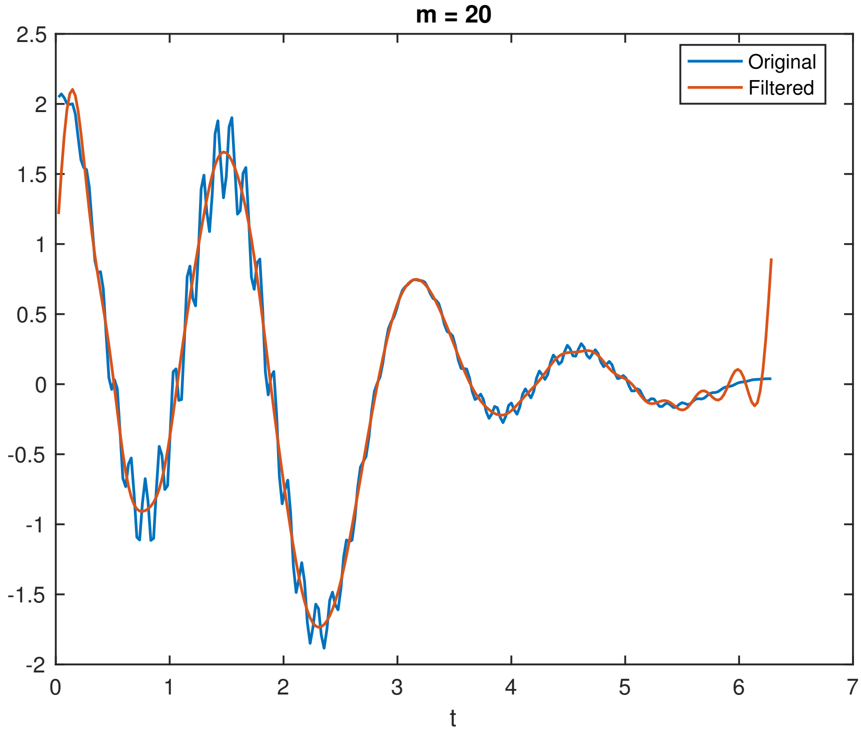
$$\hat{x}(z) = \sum_{j=-\infty}^{\infty} x_j z^{-j} = \sum_{j=0}^{\infty} \left(\frac{z}{2}\right)^j = \frac{1}{1-\frac{z}{2}} = \frac{1-\overline{z}}{\frac{5}{4}-\mathrm{Re}(z)}.$$

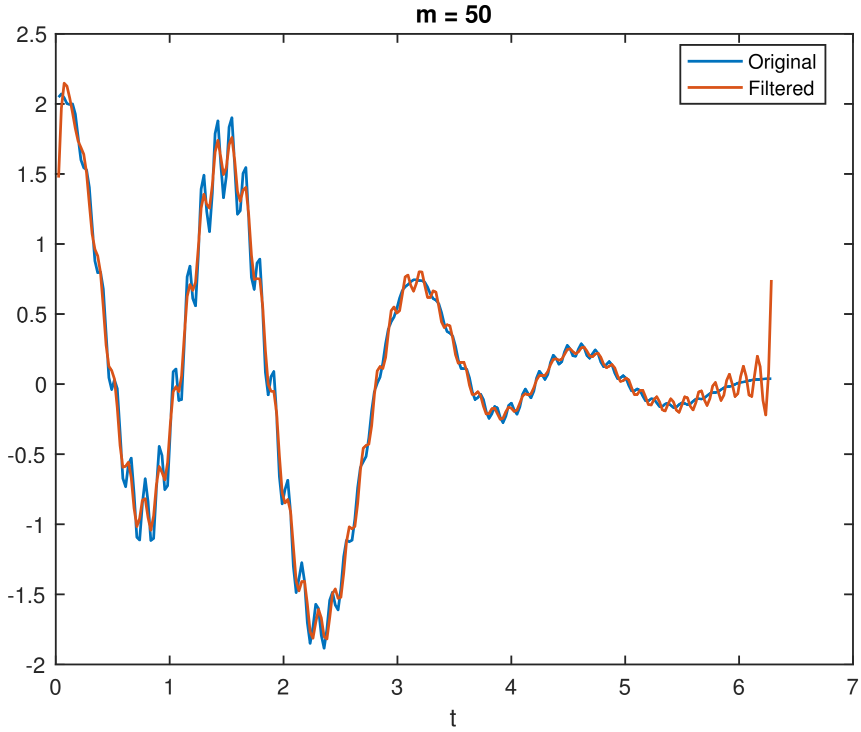
4 **B&N: 3.7** The figures below show f and the filtered versions of f with varying values of m. Two observations can be made. The first observation is that it seems as though $m = 12$ or $m = 20$ gives the best filtered signal, with good noise damping in addition to reasonable fidelity close to the edges of the interval. The second is the Gibbs-esque phenomenon occurring at the edges; the DFT assumes a signal is periodic, and the filtered signal reflects this, as it attempts to reconstruct a periodic signal, resulting in the artifacts at the ends of the interval.




m = 6
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4
5
6
7
t
Original
Filtered


m = 12
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4
5
6
7
t
Original
Filtered


m = 20
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4
5
6
7
t
Original
Filtered


m = 50
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4
5
6
7
t
Original
Filtered