Norwegian University of Science and Technology
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TMA4170 Fourier Analysis
Spring 2017

Exercise set 5 solution

1 **B&N: 3.3** We want to solve the differential equation

$$au'' + bu' + cu = f,$$

where $f(t)$ is the impulse function

$$f(t) = \begin{cases} \frac{f_0}{2h}, & -h \le t \le h \\ 0, & \text{otherwise,} \end{cases}$$

taken in the limit $h \to 0$. (Strictly speaking, this definition of f makes sense only when considering it as a *regular distribution*, i.e. it can only be applied within an integral, but we shall not dig deeper into the details of this here). We begin by taking the Fourier transform of both sides of the ODE. Since $\mathscr{F}(y') = i\lambda\mathscr{F}(y)$, we get

$$-a\lambda^2\hat{u} + ib\lambda\hat{u} + c\hat{u} = \hat{f}$$

To find $\hat{f}(\lambda)$, we use the definition of the Fourier transform and take the limit as $h \to 0$:

$$\begin{aligned}
\hat{f}(\lambda) &= \frac{f_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\lambda t} \mathrm{d}t \\
&= \lim_{h\to 0} \frac{f_0}{\sqrt{2\pi}} \int_{-h}^{h} \frac{1}{2h} e^{-i\lambda t} \mathrm{d}t \\
&= \lim_{h\to 0} \frac{f_0}{\sqrt{2\pi}} \left[\frac{-1}{2ih\lambda} e^{-i\lambda t} \right]_{-h}^{h} \\
&= \frac{f_0}{\sqrt{2\pi}\lambda} \lim_{h\to 0} \frac{\sin(\lambda h)}{h} \\
&= \frac{f_0}{\sqrt{2\pi}\lambda} \lambda \\
&= \frac{f_0}{\sqrt{2\pi}}.
\end{aligned}$$

Thus, we have

$$\hat{u}(\lambda) = \frac{f_0}{\sqrt{2\pi}} \left(\frac{1}{-a\lambda^2 + ib\lambda + c} \right).$$

We find the zeros of the polynomial on the right hand side as

$$\lambda_\pm = \frac{ib}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a} = i\mu \pm \omega,$$

where $\mu = \frac{b}{2a}$ and $\omega = \frac{\sqrt{4ac-b^2}}{2a}$. We can then factorize and split the polynomial into partial fractions to find

$$\hat{u}(\lambda) = \frac{f_0}{\sqrt{2\pi}2\omega a}\left(\frac{1}{-\lambda+\omega+i\mu} - \frac{1}{-\lambda-\omega+i\mu}\right).$$

We could now use the inverse Fourier transform directly, which would require a contour integral. However, it is easier to consult a table of Fourier transforms (keeping in mind that tables may use different conventions for the 2π factor(s) in the transform) and find that if

$$g(t) = \begin{cases} e^{-\beta t}e^{i\gamma t}, & t \geq 0 \\ 0, & t < 0, \end{cases}$$

then

$$\hat{g}(\lambda) = \frac{1}{\sqrt{2\pi}}\frac{i}{-\lambda+\gamma+i\beta}.$$

We can thus recognise that

$$u(t) = \begin{cases} \frac{f_0}{2\omega a}e^{-\mu t}\left(\frac{e^{i\omega t}-e^{-i\omega t}}{i}\right), & t \geq 0 \\ 0, & t < 0, \end{cases}$$

that is,

$$u(t) = \begin{cases} \frac{f_0}{\omega a}e^{-\mu t}\sin(\omega t), & t \geq 0 \\ 0, & t < 0. \end{cases}$$

2 **B&N: 3.10** The difference approximations to $u'(t_k)$ and $u''(t_k)$ are

$$u'(t_k) \simeq \frac{u_k - u_{k-1}}{h}, \quad u''(t_k) \simeq \frac{u_{k+1} - 2u_k + u_{k-1}}{h^2}.$$

Inserting these into the differential equation evaluated at time t_k gives

$$a\frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} + b\frac{u_k - u_{k-1}}{h} + cu_k = f_k$$
$$\Rightarrow au_{k+1} + (ch^2 + bh - 2a)u_k + (a - bh)u_{k-1} = h^2 f_k,$$

which is what we wanted to show.

3 **B&N: 3.11** Let us consider the hint given in the exercise first. We want to show that there can be no solution z with $|z| = 1$ to the quadratic equation

$$az^2 + \beta z + \gamma = 0, \tag{1}$$

where $\beta = ch^2 + bh - 2a$, $\gamma = a - bh$, and $a, b, c, h > 0$. We shall show this by contradiction. Assume that there does exist a z solving this such that $|z| = 1$. Then, $|z|^2 = z\overline{z} = 1$, so multiplying (1) by $\overline{z}$ yields

$$\begin{aligned} 0 &= az + \beta + \gamma\overline{z} \\ &= az + \beta + a\overline{z} - bh\overline{z} \\ &= (2a - bh)\mathrm{Re}(z) + \beta + ibh\mathrm{Im}(z). \end{aligned}$$

We must have equality for the real and imaginary part of this equation. Since $a, b, c, h \in \mathbb{R}$, we also have $\beta \in \mathbb{R}$. Thus, the imaginary part gives $\mathrm{Im}(z) = 0$ since $b, h > 0$, i.e. $z = \mathrm{Re}(z) \in \mathbb{R}$, and so the problem reduces to

$$(2a - bh)z + \beta = 0.$$

Since $|z| = 1$, we must have $z = 1$ or $z = -1$. With $z = 1$, the above equation reduces to $ch^2 = 0$, which is impossible. In the case $z = -1$, we need to have

$$bh - 2a + \beta = ch^2 + 2bh - 4a = 0.$$

This does seem to be possible; for some given $b, c, h > 0$ we can choose a to satisfy the equality. Or, in other terms, given $a, b, c > 0$, $z = -1$ is a solution if one uses a step length h satisfying the quadratic equation

$$ch^2 + 2bh - 4a = 0.$$

If one avoids this step length, there is no solution to (1) with $|z| = 1$. Now, what we want to show is that

$$aw^j + \beta + \gamma \overline{w}^j \neq 0.$$

Multiplying by w^j and using that $|w| = 1$, we want to show that

$$a(w^j)^2 + \beta w^j + \gamma \neq 0.$$

Since we know that $|w^j| = 1$, we know that this holds as long as $ch^2 + 2bh - 4a \neq 0$.

4 **B&N: 3.12** Taking the DFT of both sides of equation (3.11) we find that

$$a\hat{y}_{k+1} + \beta \hat{y}_k + \gamma \hat{y}_{k-1} = h^2 \hat{f}_k.$$

Next, we use the shift property of the DFT ($\hat{y}_{j+1} = w^j \hat{y}_k$), and the fact that $w^{-1} = \overline{w}$ to find

$$aw^k \hat{y}_k + \beta \hat{y}_k + \gamma \overline{w}^k \hat{y}_k = h^2 \hat{f}_k,$$

that is,

$$\hat{y}_k = h^2 \frac{\hat{f}_k}{aw^k + \beta + \gamma \overline{w}^k}.$$