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1 **B&N: 2.8** Since L is linear and time invariant, it is characterized by its impulse response function h:

$$Lf(t) = (f * h)(t).$$

If L is causal, then $Lf(T) = 0$ if $f(t) = 0$ for all $t < T$. We want to show that if L is causal and h is continuous, then $h(t) = 0$ for all $t < 0$. Following the hint, we set up a proof by contradiction; assume that $h(t_0) \neq 0$ for some $t_0 < 0$. Without loss of generality, we assume $h(t_0) > 0$. Then, by continuity, there exists a $\delta > 0$ such that $h(t) > 0$ on $[t_0 - \delta, t_0 + \delta]$. Applying L to the signal

$$f(t) = \begin{cases} 1, & 0 \le t \le \delta \\ 0, & \text{otherwise} \end{cases}$$

yields

$$Lf(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\mathrm{d}\tau = \int_{0}^{\delta} h(t-\tau)\mathrm{d}\tau = \int_{t-\delta}^{t} h(u)\mathrm{d}u,$$

such that

$$Lf(t_0) = \int_{t_0-\delta}^{t_0} h(u)\mathrm{d}u > 0.$$

Since $t_0 < 0$, L is not causal and so we have a contradiction, which means the assumption that $h(t_0) \neq 0$ for some $t_0 < 0$ must be false.

2 **B&N: 2.9** Assume $f' \in L^2$ and $\lim_{t\to\pm\infty} (t-a)|f(t)|^2 = 0$. Using integration by parts, we get

$$\begin{aligned}
\left\langle \left(\frac{\mathrm{d}}{\mathrm{d}t} - i\alpha\right)(t-a)f(t), f(t) \right\rangle &= \int_{-\infty}^{\infty} \left(\frac{\mathrm{d}}{\mathrm{d}t} - i\alpha\right)\left[(t-a)f(t)\right]\overline{f(t)}\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}\left[(t-a)f(t)\right]\overline{f(t)}\mathrm{d}t + \int_{-\infty}^{\infty} -i\alpha\left[(t-a)f(t)\right]\overline{f(t)}\mathrm{d}t \\
&= \left[(t-a)|f(t)|^2\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (t-a)f(t)\frac{\mathrm{d}}{\mathrm{d}t}\overline{f(t)}\mathrm{d}t + \int_{-\infty}^{\infty} (t-a)f(t)\overline{i\alpha f(t)}\mathrm{d}t \\
&= \int_{-\infty}^{\infty} (t-a)f(t)\overline{\left(-\frac{\mathrm{d}}{\mathrm{d}t} + i\alpha\right)f(t)}\mathrm{d}t \\
&= \left\langle (t-a)f(t), \left(-\frac{\mathrm{d}}{\mathrm{d}t} + i\alpha\right)f(t) \right\rangle.
\end{aligned}$$

Here, we have also used that $\frac{\mathrm{d}}{\mathrm{d}t}\overline{f(t)} = \overline{\frac{\mathrm{d}}{\mathrm{d}t}f(t)}$ for $f:\mathbb{R}\to\mathbb{C}$. The second term is easier. We have

$$\left\langle (t-a)\left(\frac{\mathrm{d}}{\mathrm{d}t} - i\alpha\right)f(t), f(t)\right\rangle = \int_{-\infty}^{\infty} (t-a)\left(\frac{\mathrm{d}}{\mathrm{d}t} - i\alpha\right)f(t)\overline{f(t)}\mathrm{d}t$$
$$= \int_{-\infty}^{\infty} \left(\frac{\mathrm{d}}{\mathrm{d}t} - i\alpha\right)f(t)\overline{(t-a)f(t)}\mathrm{d}t$$
$$= \left\langle \left(\frac{\mathrm{d}}{\mathrm{d}t} - i\alpha\right)f(t), (t-a)f(t)\right\rangle,$$

since $(t-a)$ is real-valued. In total, this means that

$$\left\langle \left(\frac{\mathrm{d}}{\mathrm{d}t} - i\alpha\right)(t-a)f(t), f(t)\right\rangle - \left\langle (t-a)\left(\frac{\mathrm{d}}{\mathrm{d}t} - i\alpha\right)f(t), f(t)\right\rangle$$
$$= \left\langle (t-a)f(t), \left(-\frac{\mathrm{d}}{\mathrm{d}t} + i\alpha\right)f(t)\right\rangle - \left\langle \left(\frac{\mathrm{d}}{\mathrm{d}t} - i\alpha\right)f(t), (t-a)f(t)\right\rangle.$$

3 **B&N: 2.13** Given an f such that $\hat{f}(\lambda) = 0$ outside the interval $[\omega_1, \omega_2]$, we define

$$g(t) = e^{-i\frac{\omega_1+\omega_2}{2}t} f(t),$$

and observe that

$$\hat{g}(\lambda) = \int_{-\infty}^{\infty} f(t)e^{-i(\frac{\omega_1+\omega_2}{2}+\lambda)t} = \hat{f}\left(\frac{\omega_1+\omega_2}{2} + \lambda\right).$$

Since $\hat{g}\left(-\frac{\omega_2-\omega_1}{2}\right) = \hat{f}(\omega_1)$ and $\hat{g}\left(\frac{\omega_2-\omega_1}{2}\right) = \hat{f}(\omega_2)$, we know that g is band-limited with maximum frequency $\Omega = \frac{\omega_2-\omega_1}{2}$. We can therefore use the Sampling Theorem to find

$$e^{-i\frac{\omega_1+\omega_2}{2}t} f(t) = g(t) = \sum_{j=-\infty}^{\infty} g\left(\frac{j\pi}{\Omega}\right)\frac{\sin(\Omega t - j\pi)}{\Omega t - j\pi}.$$

Rearranging terms and inserting the value for Ω, this gives

$$f(t) = e^{i\frac{\omega_1+\omega_2}{2}t} \sum_{j=-\infty}^{\infty} f\left(\frac{2j\pi}{\omega_2-\omega_1}\right) e^{-i\left(\frac{\omega_1+\omega_2}{\omega_2-\omega_1}\right)j\pi} \frac{\sin(\frac{\omega_2-\omega_1}{2}t - j\pi)}{\frac{\omega_2-\omega_1}{2}t - j\pi}.$$

4 **B&N: 2.14**

a) Following the proof of the Sampling Theorem, we expand $\hat{f}(\lambda)$ in a Fourier series on the interval $[-a\Omega, a\Omega]$ and find

$$\hat{f}(\lambda) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi\lambda}{a\Omega}}, \qquad c_n = \frac{1}{2a\Omega}\int_{-a\Omega}^{a\Omega} \hat{f}(\lambda)e^{-i\frac{n\pi\lambda}{a\Omega}}\mathrm{d}\lambda.$$

Since $a > 1$ and $\hat{f}(\lambda) = 0$ for $|\lambda| \geq \Omega$, we also have $\hat{f}(\lambda) = 0$ for $|\lambda| \geq a\Omega$. Therefore, we can increase the limits of integration in the expression for c_n to find

$$\begin{aligned} c_n &= \frac{1}{2a\Omega} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{-i\frac{n\pi\lambda}{a\Omega}} \mathrm{d}\lambda \\ &= \frac{\pi}{\sqrt{2\pi}a\Omega} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{-i\frac{n\pi\lambda}{a\Omega}} \mathrm{d}\lambda \\ &= \frac{\pi}{\sqrt{2\pi}a\Omega} f\left(-\frac{n\pi}{a\Omega}\right), \end{aligned}$$

where we have used the Fourier inversion theorem in the last equality. Changing the sign of n, we find that

$$\hat{f}(\lambda) = \sum_{n=-\infty}^{\infty} c_{-n} e^{-i\frac{n\pi\lambda}{a\Omega}}, \qquad c_{-n} = \frac{\pi}{\sqrt{2\pi}a\Omega} f\left(\frac{n\pi}{a\Omega}\right).$$

b) Looking at the figure, we can see that

$$\hat{g}_a(\lambda) = \begin{cases} \frac{\lambda + a\Omega}{(a-1)\Omega}, & -a\Omega \leq \lambda \leq -\Omega \\ 1, & -\Omega \leq \lambda \leq \Omega \\ \frac{\lambda - a\Omega}{(1-a)\Omega}, & \Omega \leq \lambda \leq a\Omega. \end{cases}$$

Applying the inverse Fourier transform, we get

$$\begin{aligned} g_a(t) &= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{(a-1)\Omega} \int_{-a\Omega}^{-\Omega} (\lambda + a\Omega) e^{i\lambda t} \mathrm{d}\lambda + \int_{-\Omega}^{\Omega} e^{i\lambda t} \mathrm{d}\lambda - \frac{1}{(a-1)\Omega} \int_{\Omega}^{a\Omega} (\lambda - a\Omega) e^{i\lambda t} \mathrm{d}\lambda \right) \\ &= \frac{1}{\sqrt{2\pi}} \left(\int_{-\Omega}^{\Omega} e^{i\lambda t} \mathrm{d}\lambda - \frac{1}{(a-1)\Omega} \int_{\Omega}^{a\Omega} (\lambda - a\Omega)(e^{i\lambda t} + e^{-i\lambda t}) \mathrm{d}\lambda \right) \\ &= \frac{1}{\sqrt{2\pi}} \left(\int_{-\Omega}^{\Omega} e^{i\lambda t} \mathrm{d}\lambda - \frac{2}{(a-1)\Omega} \int_{\Omega}^{a\Omega} (\lambda - a\Omega) \cos(\lambda t) \mathrm{d}\lambda \right) \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{t} \sin(\Omega t) - \frac{2}{(a-1)\Omega} \left(\frac{(a-1)\Omega}{t} \sin(\Omega t) + \frac{\cos(a\Omega t) - \cos(\Omega t)}{t^2} \right) \right) \\ &= \frac{\sqrt{2}(\cos(\Omega t) - \cos(a\Omega t))}{\sqrt{\pi}(a-1)\Omega t^2}. \end{aligned}$$

c) Since $\hat{f}(\lambda) = 0$ for $\lambda > 0$, we have $\hat{f}(\lambda) = \hat{f}(\lambda)\hat{g}_a(\lambda)$, that is,

$$\hat{f}(\lambda) = \sum_{n=-\infty}^{\infty} \frac{\pi}{\sqrt{2\pi}a\Omega} f\left(\frac{n\pi}{a\Omega}\right) e^{-i\frac{n\pi\lambda}{a\Omega}} \hat{g}_a(\lambda).$$

Since $\hat{g}_a(\lambda) = 0$ for $|\lambda > a\Omega|$, we find the inverse Fourier transform of this as

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-a\Omega}^{a\Omega} \left(\sum_{n=-\infty}^{\infty} \frac{\pi}{\sqrt{2\pi}a\Omega} f\left(\frac{n\pi}{a\Omega}\right) e^{-i\frac{n\pi\lambda}{a\Omega}} \hat{g}_a(\lambda) \right) e^{i\lambda t} \mathrm{d}\lambda.$$

Under the assumptions of the sampling theorem ($\hat{f}$ piecewise smooth and continuous), the Fourier sum converges uniformly and so we can interchange the integration and summation to find

$$
\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} \frac{\pi}{\sqrt{2\pi}\,a\Omega} f\left(\frac{n\pi}{a\Omega}\right)\left(\frac{1}{\sqrt{2\pi}}\int_{-a\Omega}^{a\Omega} e^{-i\frac{n\pi\lambda}{a\Omega}}\,\hat{g}_a(\lambda)e^{i\lambda t}\mathrm{d}\lambda\right)\\
&= \sum_{n=-\infty}^{\infty} \frac{\pi}{\sqrt{2\pi}\,a\Omega} f\left(\frac{n\pi}{a\Omega}\right)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\frac{n\pi\lambda}{a\Omega}}\,\hat{g}_a(\lambda)e^{i\lambda t}\mathrm{d}\lambda\right)\\
&= \sum_{n=-\infty}^{\infty} \frac{\pi}{\sqrt{2\pi}\,a\Omega} f\left(\frac{n\pi}{a\Omega}\right)\mathscr{F}^{-1}\left(e^{-i\frac{n\pi\lambda}{a\Omega}}\,\hat{g}_a(\lambda)\right)(t)\\
&= \sum_{n=-\infty}^{\infty} \frac{\pi}{\sqrt{2\pi}\,a\Omega} f\left(\frac{n\pi}{a\Omega}\right) g_a\left(t-\frac{n\pi}{a\Omega}\right).
\end{aligned}
$$