Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4170 Fourier Analysis
Spring 2017

Exercise set 3 solution

1 **B&N: 2.4** We use the definition of the Fourier transform and Euler's formula to write:

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\lambda t} \mathrm{d}t = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(\lambda t) f(t) \mathrm{d}t - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin(\lambda t) f(t) \mathrm{d}t.$$

If f is real-valued, then both integrands are real-valued. If, in addition, f is even, then $\sin(\lambda t) f(t)$ is odd, so the second integral is zero, i.e. $\hat{f}(\lambda)$ is real-valued. On the other hand, if f is odd, then $\cos(\lambda t) f(t)$ is odd, so the first integral is zero, i.e. $\hat{f}(\lambda)$ is pure imaginary.

2 **B&N: 2.5** The convolution is defined as:

$$(\phi * \phi)(x) = \int_{-\infty}^{\infty} \phi(x-t)\phi(t) \mathrm{d}x.$$

Observe that $\phi(x-t) = 1$ for $0 \le x-t < 1$, i.e. $x-1 < t \le x$, and 0 otherwise. Since $\phi(t)$ is nonzero when $0 \le t < 1$ the product $\phi(x-t)\phi(t)$ (and thus the convolution) is nonzero only when the intervals $(x-1, x]$ and $[0, 1)$ overlap, i.e. when $0 \le x < 2$. We can identify two cases; one is when $0 \le x \le 1$, when the overlap interval is $[0, x]$, and the convolution:

$$(\phi * \phi)(x) = \int_0^x \mathrm{d}x = x.$$

In the second case, we have $1 < x < 2$, which gives the overlap interval $[x-1, 1)$, and the convolution:

$$(\phi * \phi)(x) = \int_{x-1}^{1} \mathrm{d}x = 2 - x.$$

Since $1 - |x-1| = x$ when $x < 1$ and $1 - |x-1| = 2 - x$ when $x > 1$, the convolution can be summarized as

$$(\phi * \phi)(x) = \begin{cases} 1 - |x-1| & \text{if} \quad 0 \le x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

3 **B&N: 2.6** Following the definition, the Fourier transform of f is

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{s} e^{-sx^2} e^{-i\lambda x} \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{s} e^{-(sx^2 + i\lambda x)} \mathrm{d}x.$$

Completing the square, we see that $sx^2 + i\lambda x = (\sqrt{s}x + \frac{i\lambda}{2\sqrt{s}})^2 + \frac{\lambda^2}{4s}$, meaning

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{s} e^{-(\sqrt{s}x + \frac{i\lambda}{2\sqrt{s}})^2} e^{-\frac{\lambda^2}{4s}} \mathrm{d}x = \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{4s}} \int_{-\infty}^{\infty} \sqrt{s} e^{-(\sqrt{s}x + \frac{i\lambda}{2\sqrt{s}})^2} \mathrm{d}x.$$

We can then perform a change of variables $y = \sqrt{s}x$ to find that

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{4s}} \int_{-\infty}^{\infty} e^{-(y + \frac{i\lambda}{2\sqrt{s}})^2} \mathrm{d}y.$$

What remains to show is that

$$\int_{-\infty}^{\infty} e^{-(y + \frac{i\lambda}{2\sqrt{s}})^2} \mathrm{d}y = \int_{-\infty}^{\infty} e^{-y^2} \mathrm{d}y = \sqrt{\pi}, \tag{1}$$

which can seen by means of a contour integral in the complex plane. Consider integrating $g(z) = e^{-z^2}$ counterclockwise along the border of the rectangle with real values in $[-l, l]$ and imaginary values in $[0, \frac{i\lambda}{2\sqrt{s}}]$. Since g is holomorphic, the contour integral is zero by Cauchy's integral theorem, , i.e.

$$\int_{-l}^{l} e^{-(t + \frac{i\lambda}{2\sqrt{s}})^2} \mathrm{d}t + \int_{\frac{\lambda}{2\sqrt{s}}}^{0} e^{-(l+it)^2} \mathrm{d}t + \int_{l}^{-l} e^{-t^2} \mathrm{d}t + \int_{0}^{\frac{\lambda}{2\sqrt{s}}} e^{-(-l+it)^2} \mathrm{d}t = 0.$$

Taking the limit as $l \to \infty$, we use the Riemann-Lebesgue theorem and find that

$$\begin{aligned}
0 &= \int_{-\infty}^{\infty} e^{-(t + \frac{i\lambda}{2\sqrt{s}})^2} \mathrm{d}t + \lim_{l \to \infty} \int_{\frac{\lambda}{2\sqrt{s}}}^{0} e^{-l^2 + t^2} e^{-2ilt} \mathrm{d}t + \int_{\infty}^{-\infty} e^{-t^2} \mathrm{d}t + \lim_{l \to \infty} \int_{0}^{\frac{\lambda}{2\sqrt{s}}} e^{-l^2 + t^2} e^{2ilt} \mathrm{d}t \\
&= \int_{-\infty}^{\infty} e^{-(t + \frac{i\lambda}{2\sqrt{s}})^2} \mathrm{d}t + 0 + \int_{\infty}^{-\infty} e^{-t^2} \mathrm{d}t + 0 \\
&= \int_{-\infty}^{\infty} e^{-(t + \frac{i\lambda}{2\sqrt{s}})^2} \mathrm{d}t - \int_{-\infty}^{\infty} e^{-t^2} \mathrm{d}t.
\end{aligned}$$

Thus, (1) is verified, and so our proof is concluded.

4 **B&N: 2.10** Calculating the Fourier transform of h, we find

$$\hat{h}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} A e^{-\alpha t} e^{-i\lambda t} \mathrm{d}t = \frac{A}{\sqrt{2\pi}} \left[\frac{e^{-(\alpha + i\lambda)t}}{\alpha + i\lambda} \right]_0^{\infty} = \frac{A}{\sqrt{2\pi}(\alpha + i\lambda)}.$$

That $\hat{h}(\lambda) = \frac{1}{\sqrt{2\pi}} (\mathscr{L}h)(i\lambda)$ follows from property 8 in Theorem 2.6.

5 **B&N: 2.11** When computing $f * h$, we start by noticing that $f(x) = 0$ when $x < 0$ and $x > \pi$, such that

$$(f * h)(t) = \int_{-\infty}^{\infty} f(x)h(t-x)\mathrm{d}x = \int_{0}^{\pi} f(x)h(t-x)\mathrm{d}x.$$

Furthermore, $h(t-x) = 0$ when $t - x < 0$, i.e. $x > t$, so the integral reduces to

$$(f * h)(t) = \int_{0}^{t} f(x)h(t-x)\mathrm{d}x$$

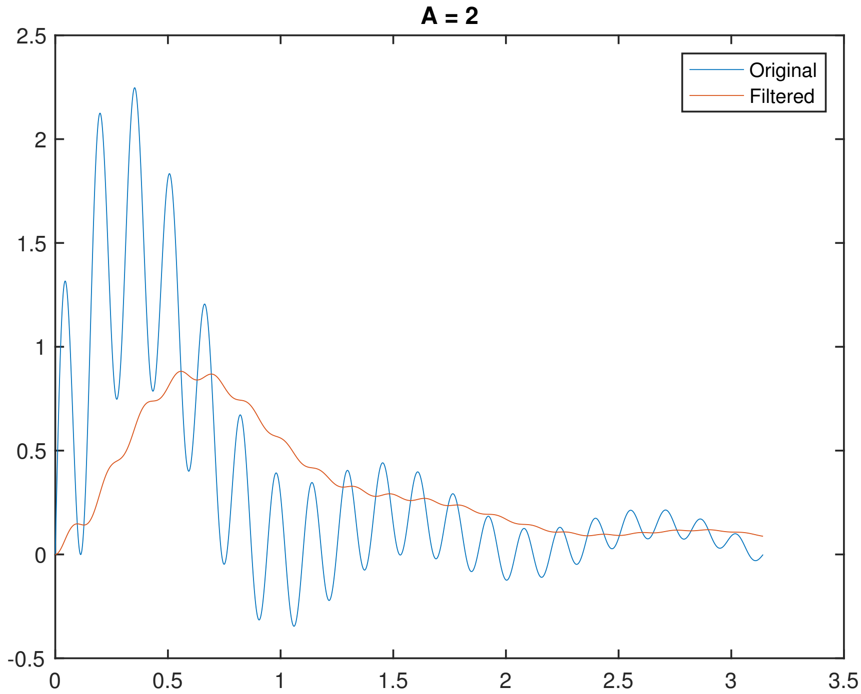
since we are not interested in values $t > \pi$. Next, using that

$$\int e^{ax}\sin(kx)\mathrm{d}x = \frac{e^{ax}(a\sin(kx) - k\cos(kx))}{a^2 + k^2} + C,$$

we find

$$\begin{aligned}
(f * h)(t) &= \int_{0}^{t} e^{-x}(\sin(5x) + \sin(3x) + \sin(x) + \sin(40x))Ae^{-\alpha(t-x)}\mathrm{d}x \\
&= Ae^{-\alpha t}\int_{0}^{t} e^{(\alpha-1)x}(\sin(5x) + \sin(3x) + \sin(x) + \sin(40x))\mathrm{d}x \\
&= Ae^{-\alpha t}\left[\frac{(\alpha-1)\sin(5t) - 5\cos(5t)}{(\alpha-1)^2 + 5^2} + \frac{(\alpha-1)\sin(3t) - 3\cos(3t)}{(\alpha-1)^2 + 3^2}\right. \\
&\qquad \left. + \frac{(\alpha-1)\sin(t) - \cos(t)}{(\alpha-1)^2 + 1} + \frac{(\alpha-1)\sin(40t) - 40\cos(40t)}{(\alpha-1)^2 + 40^2}\right] \\
&\quad + Ae^{-\alpha t}\left[\frac{5}{(1-\alpha)^2 + 5^2} + \frac{3}{(1-\alpha)^2 + 3^2} + \frac{1}{(1-\alpha)^2 + 1} + \frac{40}{(1-\alpha)^2 + 40^2}\right].
\end{aligned}$$

The following figures show plots of f and $f * h$ with differing values of $A = \alpha$. Notice how lower values of A causes the filter to remove lower frequencies from the signal.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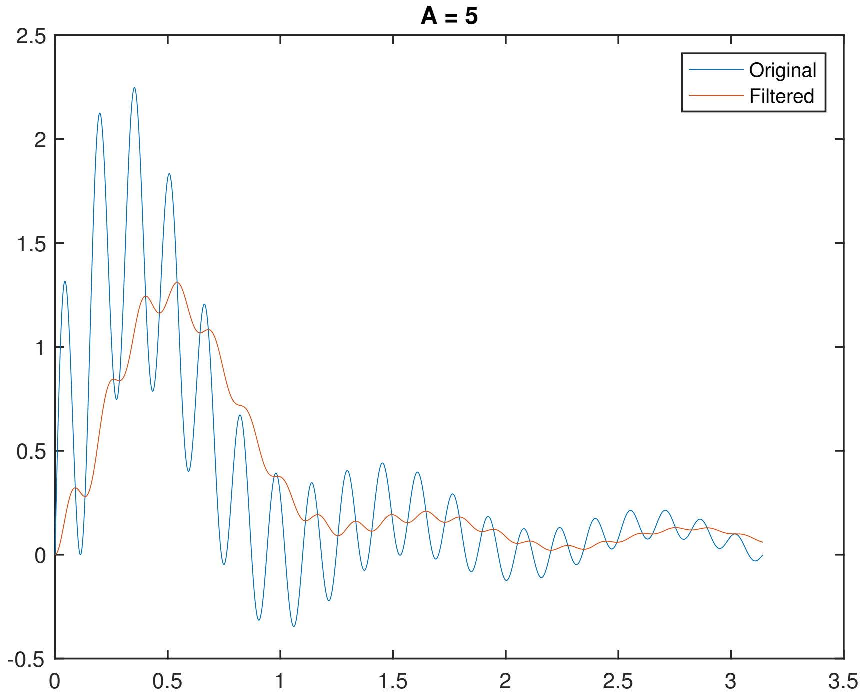
Original
Filtered
2.5
2
1.5
1
0.5
0
-0.5
0
0.5
1
1.5
2
2.5
3
3.5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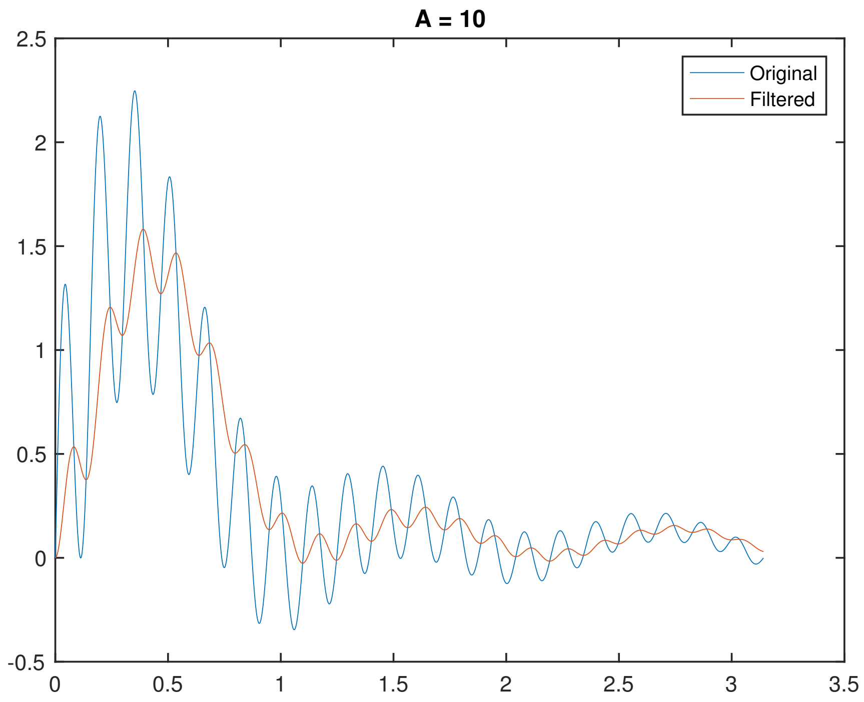
Original
Filtered
2.5
2
1.5
1
0.5
0
-0.5
0
0.5
1
1.5
2
2.5
3
3.5


A = 10
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Filtered
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1.5
1
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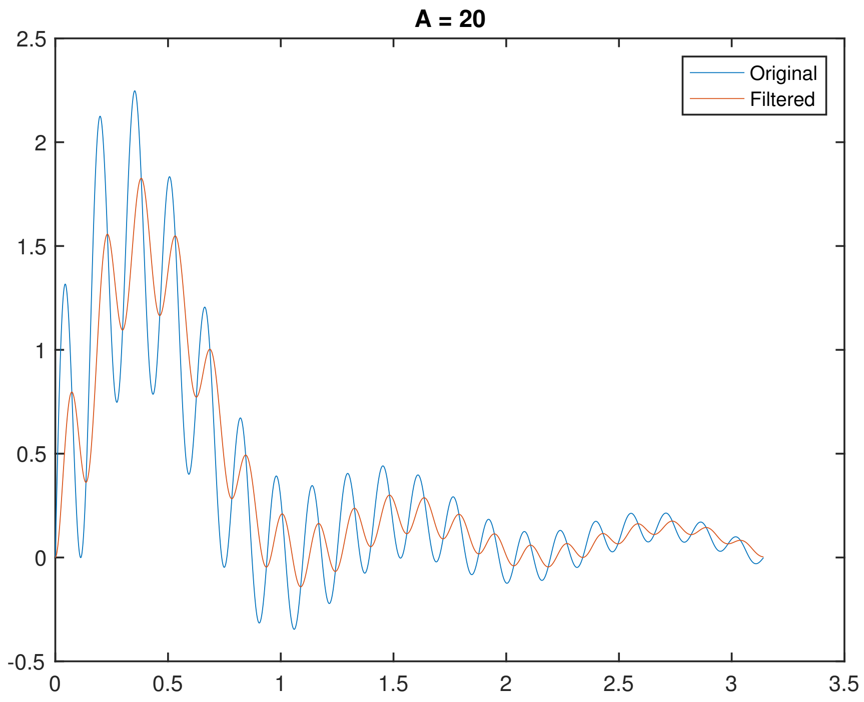
0.5
1
1.5
2
2.5
3
3.5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Filtered
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1
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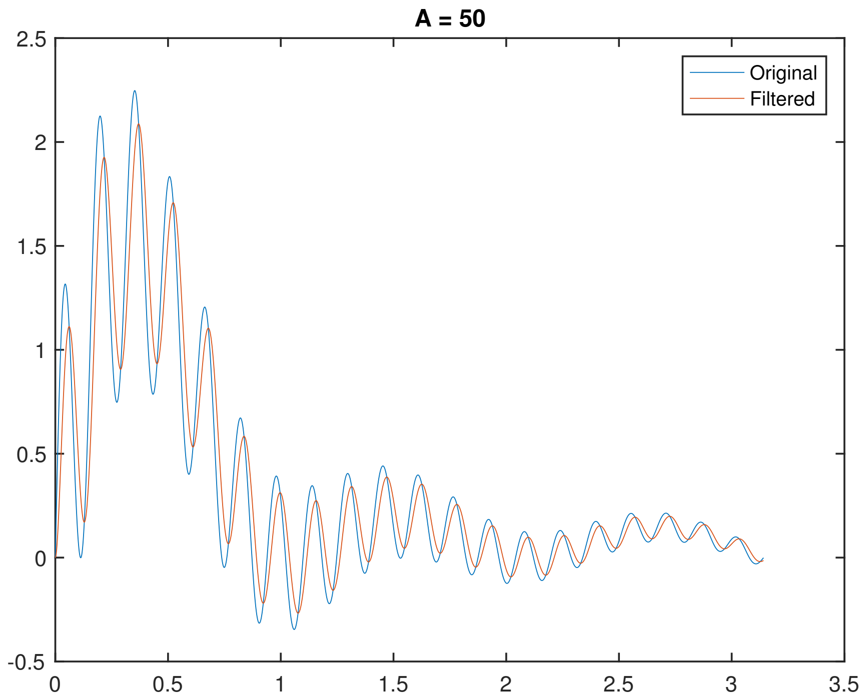
1
1.5
2
2.5
3
3.5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Original
Filtered
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1.5
1
0.5
0
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0
0.5
1
1.5
2
2.5
3
3.5