TMA4170 Fourier Analysis
Spring 2017

Exercise set 2 solution

1 **B&N: 1.1** First off, observe that $f(x) = x^2$ is even, so the Fourier series will be a cosine series. We find the coefficients:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 \mathrm{d}x = \frac{\pi^2}{3},$$
$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)\mathrm{d}x = \frac{4}{n^2}\cos(n\pi) = \frac{4}{n^2}(-1)^n,$$

so

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos(nx).$$

The sums for $N = 1, 2, 5, 7$ are shown in the figures below. Notice that on the interval $[-2\pi, 2\pi]$, the Fourier series fails, as is to be expected since it assumes the function is 2π-periodic.

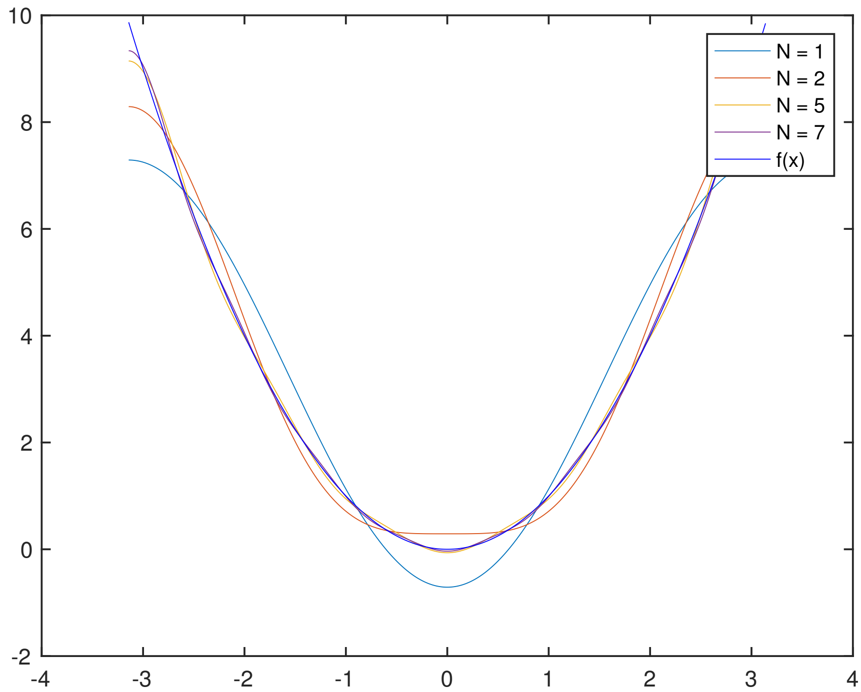

Figure 1: Convergence on $[-\pi, \pi]$

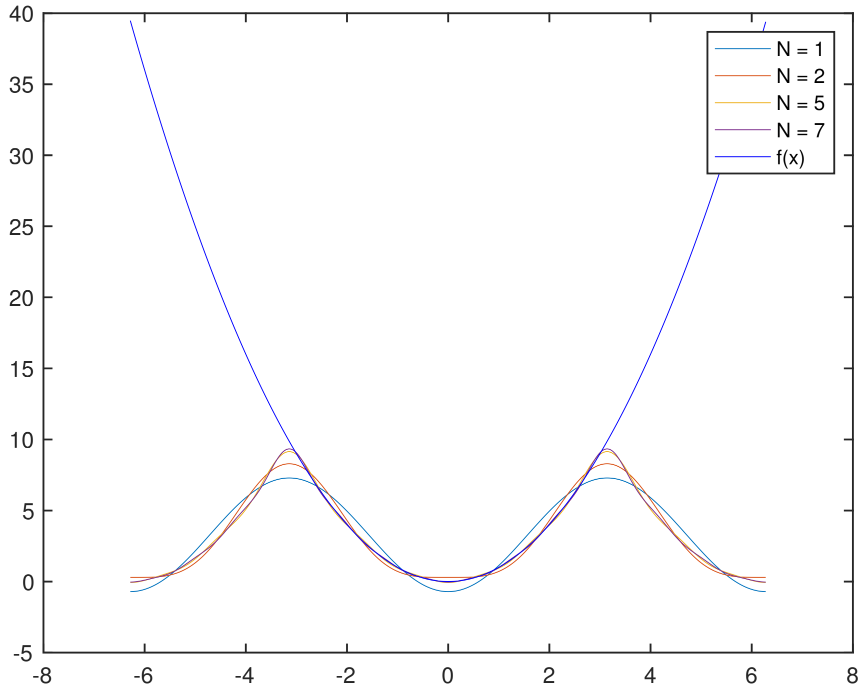

Figure 2: Convergence on $[-2\pi, 2\pi]$

2 **B&N: 1.18** With $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$ and $g(x) = \sum_{m=-\infty}^{\infty} b_m e^{imx}$, we can observe that

$$\begin{aligned}(f * g)(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(x-t)\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} a_n e^{int} \sum_{m=-\infty}^{\infty} b_m e^{im(x-t)}\mathrm{d}t \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_n b_m e^{imx}\int_{-\pi}^{\pi} e^{i(n-m)t}\mathrm{d}t \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_n b_m e^{imx} 2\pi\delta_{nm} \\ &= \sum_{n=-\infty}^{\infty} a_n b_n e^{inx},\end{aligned}$$

which is what we wanted to show.

3 **B&N: 1.20** If f is continuous on $0 \le x \le a$, then the even extension is, at least, continuous on $-a \le x < 0$. We have to check for continuity at $x = 0$. to that end, we check that the upper and lower limits coincide:

$$\lim_{x\to 0^-} f_e(x) = \lim_{x\to 0^-} f(-x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} f_e(x).$$

For the odd extension, continuity also holds on $0 \le x \le a$, but not necessarily at $x = 0$ since

$$\lim_{x\to 0^-} f_o(x) = \lim_{x\to 0^-} -f(-x) = -\lim_{x\to 0^-} f(-x) = -\lim_{x\to 0^+} f(x) = -\lim_{x\to 0^+} f_o(x).$$

The only way this holds is if $f_o(0) = 0$, i.e. $f(0) = 0$.

4 **B&N: 1.22** From Theorem 1.30, we know that the Fourier series of a continuous, piecewise smooth function converges uniformly. The function $f(x) = x^2$ is continuous and smooth on $[-\pi, \pi]$, and $f(-\pi) = f(\pi)$, i.e. f is 2π-continuous. Therefore, the Fourier series from exercise 1 converges uniformly. By evaluating f at $x = \pi$, we see that

$$\begin{aligned} f(\pi) = \pi^2 &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos(n\pi) \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \\ \Rightarrow \frac{\pi^2}{6} &= \sum_{n=1}^{\infty} \frac{1}{n^2}. \end{aligned}$$

5 **B&N: 1.25** Observe that since F is 2π-periodic and by a change of variables, we have

$$\int_{-\pi}^{-\pi+c} F(x)\mathrm{d}x = \int_{-\pi}^{-\pi+c} F(x+2\pi)\mathrm{d}x = \int_{\pi}^{\pi+c} F(y)\mathrm{d}y.$$

From this, we can see that

$$\begin{aligned} \int_{-\pi+c}^{-\pi+c} F(x)\mathrm{d}x &= \int_{-\pi+c}^{-\pi} F(x)\mathrm{d}x + \int_{-\pi}^{\pi} F(x)\mathrm{d}x + \int_{\pi}^{\pi+c} F(x)\mathrm{d}x \\ &= -\int_{-\pi}^{-\pi+c} F(x)\mathrm{d}x + \int_{-\pi}^{\pi} F(x)\mathrm{d}x + \int_{\pi}^{\pi+c} F(x)\mathrm{d}x \\ &= -\int_{\pi}^{\pi+c} F(x)\mathrm{d}x + \int_{-\pi}^{\pi} F(x)\mathrm{d}x + \int_{\pi}^{\pi+c} F(x)\mathrm{d}x \\ &= \int_{-\pi}^{\pi} F(x)\mathrm{d}x. \end{aligned}$$

6 **B&N: 1.40**

a) This is pretty straightforward. If $u(x, y) = (x + iy)^n$, then

$$u_{xx}(x,y) = \begin{cases} 0, & n < 2 \\ n(n-1)(x+iy)^{n-2}, & n \geq 2 \end{cases} \quad u_{yy}(x,y) = \begin{cases} 0, & n < 2 \\ -n(n-1)(x+iy)^{n-2}, & n \geq 2 \end{cases}$$

In any case, $u_{xx} + u_{yy} = 0$. The same goes for $u(x, y) = (x - iy)^n$.

b) Next, with $z = x+iy$, we define $S_N = \sum_{n=0}^{N} A_n z^n + A_{-n}\overline{z}^n$, and observe that by linearity of the Laplace operator,

$$\Delta S_N = \Delta \sum_{n=0}^{N} A_n z^n + A_{-n}\overline{z}^n = \sum_{n=0}^{N} A_n \Delta z^n + A_{-n}\Delta \overline{z}^n = 0,$$

since all z^n and $\overline{z}^n$ are solutions of the Laplace equation from part a). If we change to polar coordinates and write $z = re^{i\phi}$, we have

$$\begin{aligned}u(r,\phi) = \sum_{n=0}^{\infty} A_n z^n + A_{-n}\overline{z}^n &= \sum_{n=0}^{\infty} A_n r^n e^{in\phi} + A_{-n} r^n e^{-in\phi} \\ &= \sum_{n=0}^{\infty} A_n r^n e^{in\phi} + \sum_{n=-\infty}^{0} A_n r^{-n} e^{in\phi} \\ &= \sum_{n=-\infty}^{\infty} A_n r^{|n|} e^{in\phi},\end{aligned}$$

where we have taken a shortcut and written $2A_0 = A_0$.

c) With u in the above form, we see that the boundary condition reduces to

$$u(1,\phi) = \sum_{n=-\infty}^{\infty} A_n 1^{|n|} e^{in\phi} = \sum_{n=-\infty}^{\infty} A_n e^{in\phi} = f(\phi),$$

from which we see that the A_n are the Fourier coefficients of f.

d) The complex Fourier coefficients of f are given as

$$A_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)e^{-in\phi}\mathrm{d}\phi.$$

From this, we can see that since f is real valued, i.e. $\overline{f} = f$, and ϕ is real,

$$A_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)e^{in\phi}\mathrm{d}\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\overline{e^{-in\phi}}\mathrm{d}\phi = \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)e^{-in\phi}\mathrm{d}\phi} = \overline{A_n}.$$

Also, in the case $n = 0$, we see that A_0 is real valued. Now, we can see that

$$\begin{aligned}u(r,\phi) &= \sum_{n=-\infty}^{\infty} A_n r^{|n|} e^{in\phi} \\ &= \sum_{n=0}^{\infty} A_n r^n e^{in\phi} + A_{-n} r^n e^{-in\phi} - A_0 \\ &= \sum_{n=0}^{\infty} (A_n e^{in\phi} + \overline{A_n e^{in\phi}}) r^n - A_0 \\ &= \mathrm{Re}\left\{\sum_{n=0}^{\infty} 2A_n e^{in\phi} r^n - A_0\right\} \\ &= \mathrm{Re}\left\{\sum_{n=0}^{\infty} 2\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\mathrm{d}t\right) e^{in\phi} r^n - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\mathrm{d}t\right\} \\ &= \frac{1}{2\pi}\mathrm{Re}\left\{\int_{-\pi}^{\pi} f(t)\left(2\sum_{n=0}^{\infty} r^n e^{in(\phi-t)} - 1\right)\mathrm{d}t\right\}.\end{aligned}$$

e) We know that the formula is valid on the boundary $r = 1$; there, due to the A_n being the Fourier coefficients, $u(1,\phi) = f(\phi)$. On the interior, i.e. $r < 1$, we have $|r^n e^{in(\phi-t)}| = |r^n| < 1$, such that using the formula for a geometric sum is valid:

$$\sum_{n=0}^{\infty} r^n e^{in(\phi-t)} = \sum_{n=0}^{\infty} (re^{i(\phi-t)})^n = \frac{1}{1 - re^{i(\phi-t)}}.$$

Hence, we have

$$u(r,\phi) = \frac{1}{2\pi}\mathrm{Re}\left\{\int_{-\pi}^{\pi} f(t)\left(\frac{2}{1-re^{i(\phi-t)}}-1\right)\mathrm{d}t\right\}$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\mathrm{Re}\left\{\frac{2}{1-re^{i(\phi-t)}}-1\right\}\mathrm{d}t$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)P(r,\phi-t)\mathrm{d}t,$$

where

$$P(r,u) = \mathrm{Re}\left\{\frac{2}{1-re^{iu}}-1\right\}.$$

f) We start the last point by observing that

$$\frac{2}{1-re^{iu}}-1 = \frac{(1+re^{iu})(1-re^{-iu})}{(1-re^{iu})(1-re^{-iu})}$$
$$= \frac{1+re^{iu}-re^{-iu}-r^2}{1-re^{iu}-re^{-iu}+r^2}$$
$$= \frac{1-r^2+2ir\sin(u)}{1-2r\cos(u)+r^2}.$$

Taking the real part of this (note that the denominator is real), we have

$$P(r,u) = \frac{1-r^2}{1-2r\cos(u)+r^2}.$$

Using this within the formula from e), we find that

$$u(r,\phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)P(r,\phi-t)\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{1-r^2}{1-2r\cos(\phi-t)+r^2}\mathrm{d}t.$$