Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4170 Fourier Analysis Spring 2017

Exercise set 10 solution

1 B&N: 5.10

a) With

$$\phi(x) = \begin{cases} x+1, & -1 \le x \le 0 \\ 1-x, & 0 \le x \le 1 \\ 0, & |x| > 1, \end{cases}$$

we have

$$\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x) e^{-i\xi x} \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \phi(x) \left(\cos(\xi x) + i \sin(\xi x)\right) \mathrm{d}x.$$

Since ϕ is an even function, the sine integral drops out and we can consider only half the interval for the cosine term, so

$$\begin{aligned} \hat{\phi}(\xi) &= \sqrt{\frac{2}{\pi}} \int_0^1 (1-x)\cos(\xi x) \mathrm{d}x \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{1}{\xi} \sin(\xi x) - \frac{1}{\xi} x \sin(\xi x) - \frac{1}{\xi^2} \cos(\xi x) \right]_0^1 \\ &= \sqrt{\frac{2}{\pi}} \frac{1-\cos(\xi x)}{\xi^2} \\ &= 2\sqrt{\frac{2}{\pi}} \frac{\sin^2(\frac{\xi}{2})}{\xi^2}. \end{aligned}$$

b) We take the hint and differentiate (5.28) in B&N termwise:

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \frac{1}{\sin^2\left(\frac{\xi}{2}\right)} &= \sum_{k\in\mathbb{Z}} \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \frac{4}{(\xi+2\pi k)^2} \\ \Rightarrow \frac{3}{2} \frac{1}{\sin^4\left(\frac{\xi}{2}\right)} \cos^2\left(\frac{\xi}{2}\right) + \frac{1}{2} \frac{1}{\sin^2\left(\frac{\xi}{2}\right)} &= \sum_{k\in\mathbb{Z}} \frac{24}{(\xi+2\pi k)^4} \\ \Rightarrow \frac{3 - 2\sin^2\left(\frac{\xi}{2}\right)}{48 \sin^4\left(\frac{\xi}{2}\right)} &= \sum_{k\in\mathbb{Z}} \frac{1}{(\xi+2\pi k)^4}, \end{aligned}$$

where we have used $\cos(y^2) = 1 - \sin(y^2)$.

c) We take the hint once again, and see that

$$|\hat{g}(\xi+2\pi k)|^2 = \frac{8}{\pi} \frac{\sin^4\left(\frac{\xi}{2}+\pi k\right)}{(\xi+2\pi k)^4 \left(1 - \frac{2}{3}\sin^2\left(\frac{\xi}{2}+\pi k\right)\right)}$$

Note that $\sin^2\left(\frac{\xi}{2}+\pi k\right)=\sin^2\left(\frac{\xi}{2}\right)$, such that

$$\sum_{k\in\mathbb{Z}}|\hat{g}(\xi+2\pi k)|^2=\frac{8}{\pi}\frac{\sin^4\left(\frac{\xi}{2}\right)}{\left(1-\frac{2}{3}\sin^2\left(\frac{\xi}{2}\right)\right)}\sum_{k\in\mathbb{Z}}\frac{1}{(\xi+2\pi k)^4}$$
$$=\frac{8}{\pi}\frac{\sin^4\left(\frac{\xi}{2}\right)}{\left(1-\frac{2}{3}\sin^2\left(\frac{\xi}{2}\right)\right)}\frac{3-2\sin^2\left(\frac{\xi}{2}\right)}{48\sin^4\left(\frac{\xi}{2}\right)}=\frac{1}{2\pi}.$$

Thus, by Theorem 5.18, the translates $g(x-k)$ are orthonormal.

2 **B&N: 5.13** We start with a complex polynomial

$$P(z)=\frac{1}{2}\sum_{k\in\mathbb{Z}}p_k z^k$$

and define

$$Q(z)=-z\overline{P(-z)}=-z\sum_{k\in\mathbb{Z}}\overline{p_k}(-1)^k\,\overline{z}^k.$$

Assuming that $|z|=1$, we can write $z=e^{i\phi}$, such that $\overline{z}^k=e^{-ik\phi}=z^{-k}$. Thus, we have

$$Q(z)=z\overline{z}\sum_{k\in\mathbb{Z}}\overline{p_k}(-1)^{k-1}\,\overline{z}^{k-1}$$
$$=\sum_{k\in\mathbb{Z}}\overline{p_k}(-1)^{k-1}\,z^{1-k}.$$

Changing the summation variable to $m=1-k$, we get

$$Q(z)=\sum_{m\in\mathbb{Z}}\overline{p_{1-m}}(-1)^m\,z^m.$$