TMA4170 Fourier Analysis
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Norwegian University of Science and Technology
Department of Mathematical Sciences

Exercise set 1 solution

1 **B&N: 0.4** For the first part, the calculations are relatively simple. The idea is to use the definition of the inner product and then rely on the linearity of the integrals, and the fact that for a function f defined on a real domain,

$$\overline{\int_a^b f(x)\mathrm{d}x} = \overline{\int_a^b \mathrm{Re} f(x) + i\mathrm{Im} f(x)\mathrm{d}x} = \int_a^b \mathrm{Re} f(x) - i\mathrm{Im} f(x)\mathrm{d}x = \int_a^b \overline{f(x)}\mathrm{d}x.$$

Proofs of the properties are as following:

Conjugate symmetry:

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\mathrm{d}x = \int_a^b \overline{\overline{f(x)}g(x)}\mathrm{d}x = \overline{\int_a^b \overline{f(x)}g(x)\mathrm{d}x} = \overline{\langle g, f\rangle}.$$

Homogeneity:

$$\langle cf, g\rangle = \int_a^b cf(x)\overline{g(x)}\mathrm{d}x = c\int_a^b f(x)\overline{g(x)}\mathrm{d}x = c\langle f, g\rangle.$$

Bilinearity: First, we show that the inner product is additive, then show bilinearity as a consequence of the former properties.

$$\langle f+h, g\rangle = \int_a^b (f(x)+h(x))(\overline{g(x)})\mathrm{d}x = \int_a^b f(x)\overline{g(x)}\mathrm{d}x + \int_a^b h(x)\overline{g(x)}\mathrm{d}x = \langle f, g\rangle + \langle h, g\rangle.$$

Now, we can see that the inner product is linear (up to conjugation) in both arguments:

$$\langle f+ch, g\rangle = \langle f, g\rangle + \langle ch, g\rangle = \langle f, g\rangle + c\langle h, g\rangle$$
$$\langle f, g+ch\rangle = \overline{\langle g+ch, f\rangle} = \overline{\langle g, f\rangle} + \overline{c}\overline{\langle h, f\rangle} = \langle f, g\rangle + \overline{c}\langle f, h\rangle.$$

Next, we want to show the positivity property following the steps proposed in the exercise. What we want to show is that if f is a continuous function and $\int_a^b |f(t)|\mathrm{d}t = 0$, then $f(t) = 0$. This will be proved by contradiction.

Assuming that $|f(t_0)| > 0$ at some t_0, and that f is continuous, we know that for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(t_0) - f(t)| < \epsilon$ for all $t \in (t_0 - \delta, t_0 + \delta)$. In particular, taking $\epsilon = |f(t_0)|/2$, we know that there exists a $\delta > 0$ such that for all $t \in (t_0 - \delta, t_0 + \delta)$,

$$|f(t_0)| - |f(t)| \leq |f(t_0) - f(t)| < \frac{|f(t_0)|}{2},$$

i.e. $|f(t)| > |f(t_0)|/2$. Next, we use this estimate to find that

$$\int_a^b |f(t)|^2 \mathrm{d}t \geq \int_{t_0-\delta}^{t_0+\delta} |f(t)|^2 \mathrm{d}t > \int_{t_0-\delta}^{t_0+\delta} \left|\frac{f(t_0)}{2}\right|^2 \mathrm{d}t = 2\delta \frac{|f(t_0)|^2}{4} > 0,$$

which is a contradiction. Thus, our assumption that $|f(t_0)| > 0$ was wrong, meaning $f(t) = 0$ for all $t \in (a,b)$.

2 **B&N: 0.5** We wish to show that

$$\langle x, y \rangle = \sum_{n=0}^{\infty} x_n \overline{y_n}$$

defines an inner product on l^2. This is relatively straightforward, and a proof of the necessary properties is given below.

Positivity:

$$\langle x, x \rangle = \sum_{n=0}^{\infty} |x_n|^2 > 0.$$

Conjugate symmetry:

$$\langle x, y \rangle = \sum_{n=0}^{\infty} x_n \overline{y_n} = \sum_{n=0}^{\infty} \overline{\overline{x_n} y_n} = \overline{\sum_{n=0}^{\infty} \overline{x_n} y_n} = \overline{\langle y, x \rangle}.$$

Homogeneity:

$$\langle cx, y \rangle = \sum_{n=0}^{\infty} c x_n \overline{y_n} = c \sum_{n=0}^{\infty} x_n \overline{y_n} = c \langle x, y \rangle.$$

Additivity:

$$\langle x+z, y \rangle = \sum_{n=0}^{\infty} (x_n + z_n) \overline{y_n} = \sum_{n=0}^{\infty} x_n \overline{y_n} + \sum_{n=0}^{\infty} z_n \overline{y_n} = \langle x, y \rangle + \langle z, y \rangle.$$

3 **B&N: 0.11** We want to show that if a differentiable f is orthogonal to $\cos(t)$ in $L^2[0,\pi]$, then f' is orthogonal to $\sin(t)$. To this end, we use integration by parts and find that

$$0 = \int_0^{\pi} f(t)\cos(t)\mathrm{d}t = \left[f(t)\sin(t)\right]_0^{\pi} - \int_0^{\pi} f'(t)\sin(t)\mathrm{d}t = -\int_0^{\pi} f'(t)\sin(t)\mathrm{d}t.$$

4 **B&N: 0.12** We start the Gram-Schmidt orthogonalization procedure by taking $v_1 = 1$, $v_2 = x$, $v_3 = x^2$, and $v_4 = x^3$. Then, omitting calculations of norms and inner products, we

find:

$$\begin{aligned}
e_1 &= \frac{v_1}{\|v_1\|} = 1 \\
E_2 &= v_2 - \langle v_2, e_1\rangle e_1 = x - \frac{1}{2} \\
e_2 &= \frac{E_2}{\|E_2\|} = \sqrt{3}(2x-1) \\
E_3 &= v_3 - \langle v_3, e_2\rangle e_2 - \langle v_3, e_1\rangle e_1 = x^2 - x + \frac{1}{6} \\
e_3 &= \frac{E_3}{\|E_3\|} = \sqrt{5}(6x^2-6x+1) \\
E_4 &= v_4 - \langle v_4, e_3\rangle e_3 - \langle v_4, e_2\rangle e_2 - \langle v_4, e_1\rangle e_1 = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20} \\
e_4 &= \frac{E_4}{\|E_4\|} = \sqrt{7}(20x^3 - 30x^2 + 12x - 1).
\end{aligned}$$

That is, an orthogonal basis for the space spanned by $\{1, x, x^2, x^3\}$ is $\{e_1, e_2, e_3, e_4\}$.

5 **B&N: 0.17** Since $\langle u_0, v\rangle = \langle u_1, v\rangle$ for all $v \in V$, we know that $\langle u_0 - u_1, v\rangle = 0$ for all $v \in V$, and in particular,

$$\langle u_0 - u_1, u_0 - u_1\rangle = 0.$$

By positivity of the inner product, we know that the only possibility is $u_0 - u_1 = 0$, meaning $u_0 = u_1$.

6 **B&N: 0.23** Let $\{v_i\}_{i=1}^N$ be a set of orthonormal vectors, i.e. $\langle v_i, v_j\rangle = \delta_{ij}$. The vectors are linearly independent if

$$\sum_{i=1}^N a_i v_i = 0 \Leftrightarrow a_i = 0, i = 1, ..., N.$$

The "$\Leftarrow$" direction is trivial; if all $a_i = 0$, then $\sum_{i=1}^N a_i v_i = 0$. For the "$\Rightarrow$" direction, we assume

$$\sum_{i=1}^N a_i v_i = 0.$$

Then, taking the inner product with any v_j and using the orthonormality property, we have

$$0 = \left\langle \sum_{i=1}^N a_i v_i, v_j \right\rangle = \sum_{i=1}^N a_i \langle v_i, v_j\rangle = a_j.$$

This holds for all j, so all a_j are zero.