Solutions TMA4170, Spring 2014

Problem 1

1a

$$\hat{\chi}_{(-1,1)}(\xi) = \int_{-1}^{1} e^{-2i\pi\xi t} dt = -\frac{1}{2i\pi\xi}\left(e^{-2i\pi\xi} - e^{2i\pi\xi}\right) = \frac{\sin 2\pi\xi}{\pi\xi}.$$

1b

$$\mathcal{F}\left[(1-t^2)\chi_{(-1,1)}(t)\right](\xi) = \hat{\chi}_{(-1,1)}(\xi) - \mathcal{F}\left[t^2)\chi_{(-1,1)}(t)\right](\xi) =$$
$$\frac{\sin 2\pi\xi}{\pi\xi} + \frac{1}{4\pi^2}\left(\frac{\sin 2\pi\xi}{\pi\xi}\right)'' = \ldots = \frac{1}{\pi^2\xi^2}\left(\frac{\sin 2\pi\xi}{2\pi\xi} - \cos 2\pi\xi\right).$$

1c. *Parseval:*

$$2 = \int_{-1}^{1} \chi_{(-1,1)(t)^2} = \int_{-\infty}^{\infty}\left(\frac{\sin 2\pi\xi}{\pi\xi}\right)^2 d\xi = \frac{2}{\pi}\int_0^{\infty}\left(\frac{\sin 2x}{x}\right)^2 dx.$$

Finally

$$\int_0^{\infty}\left(\frac{\sin 2x}{x}\right)^2 dx = \pi.$$

Problem 2

2a We have $f(t) = \cos at$, $t \in (-\pi, \pi)$.

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt.$$

Since f is even we have $b_n = 0$. Further

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos at dt = \frac{1}{a\pi}\sin a\pi.$$

In order to find a_n, $n \neq 0$ we apply the relation $\cos\alpha\cos\beta = (\cos(\alpha+\beta) + \cos(\alpha-\beta))/2$:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos at \cos nt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(a+n)tdt + \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(a+n)tdt = I_1(n) + I_2(n);$$

$$I_1(n) = \frac{1}{\pi(a+n)}\sin(a+n)\pi = \frac{(-1)^n \sin a\pi}{\pi(a+n)},\ I_2(n) = \frac{1}{\pi(a-n)}\sin(a-n)\pi = \frac{(-1)^n \sin a\pi}{\pi(a-n)}.$$

Finally

$$a_n = \frac{(-1)^n \sin a\pi}{\pi}\left(\frac{1}{a+n} - \frac{1}{a-n}\right),$$

and

$$\cos at = \frac{\sin a\pi}{\pi}\left(\frac{1}{a} + \sum_{1}^{\infty}(-1)^n\left(\frac{1}{a+n} + \frac{1}{a-n}\right)\cos nt\right).$$

2b Can be obtained from the previous relation if put $t = 0$ and $a = z$.

Problem 3

We look for a solution in the form $u = u_1 + u_2$ where u_1 and u_2 are solutions to the equations

$$\left(\frac{d^2}{dx^2} - 9\right)u_1 = e^{ix},$$

and

$$\left(\frac{d^2}{dx^2} - 9\right)u_2 = \delta.$$

The first equation is straightforward, for example the function $u_1(x) = -0.1e^{ix}$ is a solution to this equation.

The Fourier transform of the second equation gives

$$-(4\pi^2\xi^2+9)\hat{u}_2 = 1 \ \Rightarrow \hat{u}_2 = \frac{-1}{4\pi^2+9}.$$

We use the formula:

$$\mathcal{F}: e^{-a|x|} \mapsto \frac{2a}{a^2+4\pi^2\xi^2},\ a>0$$

Therefore $u_2(x) = \frac{-1}{3}e^{-3|x|}$ and a final solution

$$u(x) = e^{ix} - \frac{1}{3}e^{-3|x|}.$$

Remark 1 You may obtain another solution which differs by one obtained above by a linear combination of elementary solutions e^{3x} and e^{-3x}.

Remark 2 You actually can solve equation with respect u_2 without using the Fourier transform if thinking about combining the elementary solutions above into a continuous function which meets the homogeneous equation for $x \neq 0$ and whose first derivative has jump $+1$ at $x = 0$.

Problem 4

First see how does T act on test functions. For a test function ϕ we have

$$<T,\phi> = <x\delta',\phi> = <\delta', x\phi> = -<\delta,(x\phi)'> = \phi(0).$$

Respectively $T = \delta$ and $T' = \delta'$.

Problem 5

5a

We have $g'' + \alpha g' + g = f$. The Fourier transform yields $(2i\pi\lambda)^2 + 2i\pi\alpha\lambda + 1)\hat{g}(\lambda) = \hat{f}(\lambda)$.

Therefore

$$H(\lambda) = \frac{1}{(2i\pi\lambda)^2 + 2i\pi\alpha\lambda + 1};\ \hat{g} = H\hat{f}.$$

Let $Q(\xi) = \xi^2 + \alpha\xi + 1$. We have

$$Q(\xi) = (\xi-\xi_1)(\xi-\xi_2),\ \xi_{1,2} = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - 1}.$$

Respectively

$$\frac{1}{Q(\xi)} = \begin{cases} \left(\frac{1}{\xi-\xi_1} - \frac{1}{\xi-\xi_2}\right)\frac{1}{\xi_1-\xi_2}, & \xi_1 \neq \xi_2; \\ \frac{1}{(\xi-\xi_1)^2}, & \xi_1 = \xi_2, \end{cases}$$

and, denoting

$$\lambda_{1,2} = \xi_{1,2}/2i\pi = -\frac{\alpha}{2i\pi} \pm \frac{1}{2i\pi}\sqrt{\frac{\alpha^2}{4} - 1},$$

$$H(\lambda) = \begin{cases} \frac{1}{2i\pi}\left(\frac{1}{\lambda-\lambda_1} - \frac{1}{\lambda-\lambda_2}\right)\frac{1}{2\sqrt{\alpha^2/4-1}}, & \alpha \neq \pm 2; \\ \frac{1}{(2i\pi)^2}\frac{1}{(\lambda-\lambda_1)^2}, & \alpha = \pm 2. \end{cases}$$

We apply the formula

$$\mathcal{F}^{-1}\left(\frac{1}{2i\pi}\frac{1}{\lambda - a}\right) = \begin{cases} e^{2i\pi a t}u(t), & \Im a > 0, \\ -e^{2i\pi a t}u(-t), & \Im a < 0, \end{cases}$$

here u is the Heaviside function.

Further

$$\Im\lambda_{1,2} = \begin{cases} \frac{\alpha}{4\pi}, & |\alpha| \leq 2; \\ \frac{1}{2\pi}\left(\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2-1}{4}}\right), & |\alpha| \geq 2. \end{cases}$$

We observe that

$$\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - 1} > 0 \text{ if } \alpha > 2, \quad \text{and} \quad \frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - 1} < 0 \text{ if } \alpha < 0.$$

so

$$h(t) = \frac{1}{2\sqrt{\frac{\alpha^2}{4} - 1}}\left(e^{\xi_1 t} - e^{\xi_2 t}\right)u(t),\ \text{for } t > 0, t \neq 2,$$

$$h(t) = -\frac{1}{2\sqrt{\frac{\alpha^2}{4} - 1}}\left(e^{\xi_1 t} - e^{\xi_2 t}\right)u(-t),\ \text{for } t < 0, t \neq -2$$

You can simplify these expression using sin and sinh functions.

Respectively for $\alpha = \pm 2$ one gets multiple routs and the solution of the form

$$h(t) = \pm t e^{\pm t} u(\mp t), \text{ for } \pm \alpha = 2.$$

5b The solution is stable and realizable for $\alpha > 0$.

Problem 6

$$\begin{aligned} f(x) = -\phi(4x) + 4\phi(4x-1) + 2\phi(4x-2) - 3\phi(4x-3) = \\ \frac{3}{2}\phi(2x) - \frac{1}{2}\phi(2x-1) - \frac{5}{2}\psi(2x) + \frac{3}{2}\psi(2x-1) = \\ \frac{1}{2}\phi(x) + \psi(x) - \frac{5}{2}\psi(2x) + \frac{3}{2}\psi(2x-1). \end{aligned}$$

One can also use normalized wavelet and scaling functions and use the standard algorithm. This would not be faster for the given case (but still correct of course).