TMA 4170 FOURIER ANALYSIS

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(1)
$$|\sin(x)| = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin^2(nx)}{4n^2-1}$$

NOT a Fourier Series!

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{2-e^{+2inx}-e^{-2inx}}{4n^2-1}$$

Now it is.

$$= \frac{2}{\pi}\left\{\underbrace{\sum_{n=1}^{\infty}\frac{2}{4n^2-1}}_{=1\ ^{*)}} - \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2inx}}{4n^2-1}\right\}$$

$$= \sum_{-\infty}^{\infty} c_n e^{inx}, \quad c_0 = \frac{2}{\pi}, \quad c_{2n} = \frac{-2/\pi}{4n^2-1}$$

Parseval's formula

$$\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2(x)\,dx}_{=1/2} = \sum_{n=-\infty}^{\infty}|c_n|^2 = \frac{4}{\pi^2}\left\{1+\underbrace{\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(\frac{1}{4n^2-1}\right)^2}_{2\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}}\right\}$$

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2} = \frac{\pi^2-8}{16}}$$

$$^{*)}\quad \frac{2}{4n^2-1} = \frac{1}{2n-1} - \frac{1}{2n+1}$$

Elementary sum!

Can also be obtained from $\int_{-\pi}^{\pi}|\sin x|\,dx = \frac{8}{\pi}\ldots\ldots$

Telescoping series

$$\sum_{n=1}^{N}\frac{2}{4n^2-1} = 1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\cdots-\frac{1}{2N+1} = 1-\frac{1}{2N+1} \longrightarrow 1$$

② MISPRINT $\frac{1}{|\bar{x}|^2}$

Remark: Indeed,

$$\widehat{\frac{1}{|\bar{x}|}} = C\,\frac{1}{|\bar{\xi}|^2} \quad \text{in } \mathbb{R}^3$$

Difficult.

Let A be an orthogonal 3×3-matrix:

$$AA^T = A^T A = I, \quad \det(A) = +1.$$

Assume $\underline{f(A\bar{x}) = f(\bar{x})}$ for all such A.

Then $\hat{f}(A\bar{\xi}) = \hat{f}(\bar{\xi})$, since

$$\hat{f}(A\bar{\xi}) = \iiint f(\bar{x})\, e^{-2\pi i \langle A^T\bar{x},\, \xi\rangle}\, d^3\bar{x}$$

$$\bar{y} = A^T\bar{x}$$
$$d^3\bar{y} = d^3\bar{x}$$
$$\bar{x} = A\bar{y}$$

$$= \iiint \underline{f(A\bar{y})}\, e^{-2\pi i\langle \bar{y},\bar{\xi}\rangle}\, d^3\bar{y}$$

$$= \iiint \underline{f(\bar{y})}\, e^{-2\pi i\langle \bar{y},\bar{\xi}\rangle}\, d^3\bar{y} = \hat{f}(\bar{\xi}).$$

Using spherical coordinates

$$\begin{cases} x_1 = r\sin\theta\cos\phi \\ x_2 = r\sin\theta\cos\phi \\ x_3 = r\cos\theta \end{cases} \qquad \bar{\xi} = (0, 0, |\bar{\xi}|)$$

NOT REQUIRED! THAT $\int_0^\infty \sin(2\pi|\xi| r)\, dr$ IS INTERPRETED

we get

$$\widehat{\frac{1}{|\bar{x}|^2}} = 2\pi \int_0^\pi\!\!\int_0^\infty \frac{r^2}{r}\, \underbrace{e^{-2\pi i|\bar{\xi}| r\cos\theta} \sin\theta\, dr\, d\theta}_{\frac{\partial}{\partial\theta}\left(\frac{e^{-2\pi i|\bar{\xi}| r\cos\theta}}{-2\pi i|\bar{\xi}| r}\right)}$$

$$= 2\pi\,\frac{1}{\pi|\bar{\xi}|}\int_0^\infty \frac{\sin(2\pi|\bar{\xi}| r)}{r}\, dr$$

DIVERGENT WITHOUT $1/r$

(3a)

$$\frac{dS_{2N}(x)}{dx} = \frac{1}{\pi}\sum_{n=1}^{N} 2\cos((2n-1)x)$$

$$= \frac{1}{\pi}\sum_{n=1}^{N}\left(e^{(2n-1)ix} + e^{-(2n-1)ix}\right)$$

$$= \frac{1}{\pi}e^{-ix}\sum_{n=1}^{N} e^{(2ix)n} + \frac{1}{\pi}e^{+ix}\sum_{n=1}^{N} e^{(-2\pi i)n}$$

[Geometric series]

$$= \cdots\cdots = \frac{\sin(2Nx)}{\sin x}\cdot\frac{1}{\pi}, \quad x_{2N} = \frac{\pi}{2N}$$

(3b)

$$S_{2N}(x_{2N}) = \frac{1}{2} + \frac{1}{\pi}\int_0^{x_{2N}} \frac{\sin(2Nx)}{\sin(x)}\,dx$$

SUBST.: $2Nx = y$

$$= \frac{1}{2} + \frac{1}{\pi}\int_0^{\pi} \frac{\sin(y)}{y}\,\frac{dy}{\left[\dfrac{\sin(y/2N)}{y/2N}\right]}.$$

$$\longrightarrow \frac{1}{2} + \frac{1}{\pi}\int_0^{\pi}\frac{\sin(y)}{y}\,dy = \frac{1}{2} + 0{,}59\ldots > 1$$

Hence $G > 1$.

④ $\hat{T}(\phi) \overset{\text{def.}}{=} T(\hat{\phi})$

$$\int_{|x|\geq\varepsilon} \phi' \ln|x|\, dx = \cdots = [\phi(-\varepsilon)-\phi(\varepsilon)]\ln(\varepsilon)$$

$$\underbrace{-\int_{\varepsilon}^{\infty} \frac{\phi(x)}{x}\,dx - \int_{-\infty}^{-\varepsilon} \frac{\phi(x)}{x}\,dx}$$

$$= -\int_{\varepsilon}^{\infty} \frac{\phi(x)-\phi(-x)}{x}\,dx$$

$$-T(\phi) = -\lim_{\varepsilon\to 0} \int_{|x|\geq\varepsilon} \phi'(x)\ln|x|\,dx = +\int_0^{\infty} \frac{\phi(x)-\phi(-x)}{x}\,dx$$

$$T(\hat{\phi}) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty}\int_{-\infty}^{\infty} \phi(x)\,\frac{e^{i\omega x}-e^{-i\omega x}}{\omega}\,dx\,d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_0^{\infty} \phi(x)\,\frac{2i\sin(\omega x)}{\omega}\,d\omega\,dx$$

$$= \frac{2i\pi}{\sqrt{2\pi}}\,\frac{1}{2}\int_{-\infty}^{\infty} \operatorname{sign}(x)\,\phi(x)\,dx$$

$$\int_{-\infty}^{\infty} \frac{\sin(\omega x)}{\omega}\,d\omega = \pm\pi.$$

Symbolically $\hat{T} = -\sqrt{\frac{\pi}{2}}\; i\,\operatorname{sign}(x)$

⑤ $2\pi \sum |\hat{g}(\omega + 2n\pi)|^2 = 1 + 2\cos(\omega)$

MISPRINT $1 + \frac{1}{2}\cos(\omega)$. The misprint has no effect on the calculations.

$$\gamma_k = \int_{-\infty}^{\infty} g(x)\,\overline{g(x-k)}\,dx = \int_{-\infty}^{\infty} \hat{g}(\omega)\,\overline{\hat{g}(\omega)}\,e^{+i\omega k}\,d\omega$$

$$= \sum \int_{2n\pi}^{2n\pi+2\pi} |\hat{g}(\omega)|^2 e^{i\omega k}\,d\omega = \sum \int_{0}^{2\pi} |\hat{g}(\xi+2n\pi)|^2 e^{i\xi k}\,d\xi$$

$$= \int_{0}^{2\pi} \sum |\hat{g}(\xi+2n\pi)|^2 e^{ik\xi}\,d\xi$$

$$= \frac{1}{2\pi}\int_{0}^{2\pi} \left(1 + e^{i\xi} + e^{-i\xi}\right) e^{ik\xi}\,d\xi$$

$\gamma_0 = 1$, $\gamma_1 = 1\ (1/4)$, $\gamma_{-1} = 1\ (1/4)$ and all other $\gamma_k = 0$.