

# THE FOURIER TRANSFORM

$$\text{sign}(x) \longleftrightarrow -\frac{i}{\pi} \text{PV}\left(\frac{1}{\xi}\right)$$

NORMALIZATION:

$$\hat{\phi}(\xi) = \int e^{-2\pi i \xi x} \phi(x) dx$$

$$1^\circ) \quad \underbrace{e^{-2\pi \varepsilon |x|} \text{sign}(x)}_{\text{A FUNCTION IN } L^1(\mathbb{R})} = -\frac{i}{\pi} \underbrace{\frac{\xi}{\xi^2 + \varepsilon^2}}_{\text{A FUNCTION IN } L^1(\mathbb{R}) \text{ (locally)}}$$

Indeed  $\int_{-\infty}^{\infty} e^{-2\pi \varepsilon |x|} \text{sign}(x) e^{-2i\pi x \xi} dx$

$$= \int_0^{\infty} e^{-2\pi x(\varepsilon + i\xi)} dx - \int_{-\infty}^0 e^{2\pi x(\varepsilon - i\xi)} dx = \dots!$$

Remark: The regularization avoids difficulties in Fubini's Theorem, which occur in  $\langle \text{sign}(x), \hat{\varphi} \rangle$ .

$$2^\circ) \quad e^{-2\pi \varepsilon |x|} \text{sign}(x) \xrightarrow[\varepsilon \rightarrow 0+]{f'} \boxed{\text{sign}(x)}$$

$$-\frac{i}{\pi} \frac{\xi}{\xi^2 + \varepsilon^2} \xrightarrow[\varepsilon \rightarrow 0+]{f'} \boxed{-\frac{i}{\pi} \text{PV}\left(\frac{1}{\xi}\right)} \approx 2\varphi'(0)\xi$$

For example,

$$\int_{-\infty}^{\infty} \frac{\xi \varphi(\xi)}{\xi^2 + \varepsilon^2} d\xi = \int_0^{\infty} \frac{\xi}{\xi^2 + \varepsilon^2} [\varphi(\xi) - \varphi(-\xi)] d\xi \xrightarrow{\varepsilon \rightarrow 0} \int_0^{\infty} \frac{\varphi(\xi) - \varphi(-\xi)}{\xi} d\xi$$

$$3^\circ) \quad T_\varepsilon \rightarrow T \quad \hat{T}_\varepsilon \rightarrow S^0 \text{ in } f' \\ \Rightarrow \hat{T} = S \quad (T(\hat{\varphi}) = S(\varphi))$$

The result follows.