§ 2.6
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Oversampling Recall the formula

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{in\pi t/d}, \quad c_n = \frac{1}{2d}\int_{-d}^{d} f(t) e^{-n\pi t/d}\, dt$$

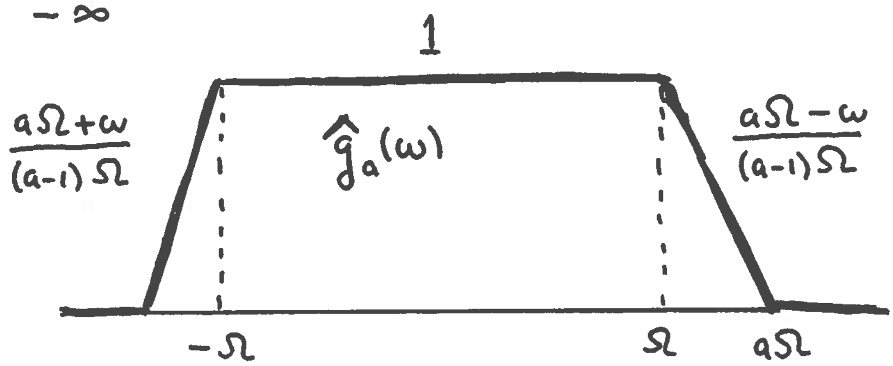
for the Fourier expansion. (The period is $= 2d$.)

a) $\hat{f}(\omega) = 0$, when $|\omega| > \Omega$. Choose $d = a\Omega$, where $a > 1$.

$$\hat{f}(\omega) = \sum_{-\infty}^{\infty} c_{-n} e^{-i\frac{n\pi\omega}{a\Omega}}; \quad c_{-n} = \frac{1}{2a\Omega}\int_{-a\Omega}^{+a\Omega} \hat{f}(\omega) e^{+\frac{in\pi\omega}{a\Omega}}\, d\omega$$

$$= \frac{1}{2a\Omega}\int_{-\infty}^{\infty} \hat{f}(\omega) e^{\frac{in\pi\omega}{a\Omega}}\, d\omega = \sqrt{\frac{\pi}{2}}\,\frac{1}{a\Omega}\, f\left(\frac{n\pi}{a\Omega}\right)$$

b)



$$g_a(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{g}_a(\omega) e^{i\omega t}\, d\omega = \cdots\cdots =$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{\cos(\Omega t) - \cos(a\Omega t)}{(a-1)\Omega\, t^2}.$$

Notice that $g_a(t) \longrightarrow \sqrt{\frac{2}{\pi}}\,\frac{\sin(\Omega t)}{t} = \sqrt{\frac{2}{\pi}}\,\Omega\,\mathrm{sinc}(\Omega t)$ and $\hat{g}_a(\omega) = \mathbb{1}_{[-\Omega,\Omega]}(\omega)$, as $a \longrightarrow 1+$. (This is the case treated in Shannons Sampling Theorem).

c) $\hat{f}(\omega) = \hat{f}(\omega)\,\hat{g}_a(\omega)$ (obvious identity!)

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}_a(\omega)\, e^{i\omega t} d\omega$$

$$\overset{a)}{=} \frac{1}{2a\Omega} \int_{-\infty}^{\infty} \left(\sum_{-\infty}^{\infty} f\left(\frac{n\pi}{a\Omega}\right) e^{-i\frac{n\pi\omega}{a\Omega}} \right) \hat{g}_a(\omega) e^{i\omega t} d\omega$$

$$= \frac{1}{2a\Omega} \sum f\left(\frac{n\pi}{a\Omega}\right) \int_{-\infty}^{\infty} \hat{g}_a(\omega) e^{i\omega\left(t - \frac{n\pi}{a\Omega}\right)} d\omega$$

$$\overset{\text{INVERSION}}{=} \sqrt{\frac{\pi}{2}} \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{a\Omega}\right) g_a\left(t - \frac{n\pi}{a\Omega}\right)$$

Thus the function $f(t)$ is sampled at the equidistant points

$$\frac{n\pi}{a\Omega} \qquad (n = 0, \pm 1, \pm 2, \dots)$$

See page 129 for a discussion.

Example $\displaystyle\int_0^{\infty} \frac{\sin(ax)}{x} dx = \frac{\pi}{2}$, when $a > 0$. Therefore

$$\frac{\pi}{2} = \lim_{a\to 0+} \int_0^{\infty} \cdots dx \neq \int_0^{\infty} \lim_{a\to 0+}(\cdots) dx. \text{ Yet, } \lim_{a\to 0+} \int_0^{100^{100}} \frac{\sin(ax)}{x} dx = 0.$$

The Dominated Convergence Theorem can be applied, since

$\left| \frac{\sin(ax)}{x} \right| \le |a| \le 1$ (for small a; recall that $a \to 0+$).

Moreover, $\displaystyle\int_0^{100^{100}} 1\, dx < \infty$. This was a finite interval.