FOURIER ANALYSIS

① $N = 4$.

$$\hat{y}_k = \sum_{j=0}^{3} y_j e^{-\frac{2\pi i}{4} jk} \qquad (k = 0, 1, 2, 3)$$

conjugate

or in matrix notation $\hat{y} = \overline{F_4}\, y$. Find the matrices

$$F_4, \ \overline{F_4}, \ \text{and} \ F_4^{-1}$$

Calculate $\det(F_4)$.

② Apply Poisson's Summation Formula

$$\sum f(2n\pi) = \frac{1}{\sqrt{2\pi}} \sum \hat{f}(n)$$

to the function $f(x) = e^{-a|x|}$, $a > 0$.

③ Verify

$$\widehat{f(ax-b)} = e^{-\frac{\omega b i}{a}} \frac{1}{|a|} \hat{f}\left(\frac{\omega}{a}\right), \qquad a \neq 0.$$

Find $\widehat{\phi(2^j x - k)} = \ ?$

④ Show that

$$\int_{-\infty}^{\infty} \frac{\phi(x) + \phi(-x) - 2\phi(0)}{2x^2}\, dx = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{x^2 - \varepsilon^2}{(x^2+\varepsilon^2)^2} \phi(x)\, dx$$

Infinitely many derivatives.

for all $\phi \in C_0^\infty(\mathbb{R})$. — This is the distribution $\langle x^{-2}, \phi \rangle$.

Compact Support

(5) The function $f = f(\bar{x}) = f(x_1, x_2, \dots, x_n)$ is radial(ly symmetric) if

$$f(A\bar{x}) \equiv f(\bar{x})$$

for all orthonormal matrices A, i.e.,

$$A^{-1} = A^T, \quad \det A = 1.$$

Show that, if f is radial, so is

$$\hat{f}(\bar{\xi}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\bar{x})\, e^{-2\pi i \langle \bar{x}, \bar{\xi} \rangle}\, d\bar{x}$$

when $d\bar{x} = dx_1 \cdots dx_n$. (— A radial function has a radial Fourier transform.)

(6) Let $f_0(r) = f(x,y,z)$ for the radial function f, $r = \sqrt{x^2+y^2+z^2}$. Verify

$$\hat{f}(\xi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\, e^{-2\pi i (x\xi_1 + y\xi_2 + z\xi_3)}\, dx\,dy\,dz$$

$$= \frac{2}{|\xi|} \int_0^{\infty} r\, f_0(r) \sin(2\pi r |\xi|)\, dr$$

where $|\xi| = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$.