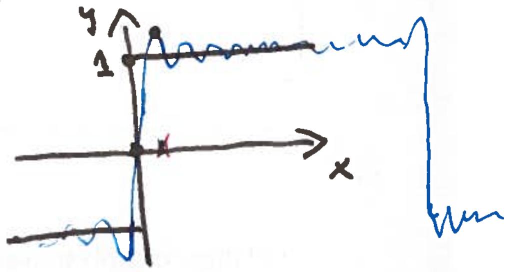
25. IV. 2016

Exercises

① Gibbs' Phenomenon for

$$f(x) = \begin{cases} 1, & 0 < x < \pi \\ -1, & -\pi < x < 0 \end{cases}$$

- Expand $f(x)$ in a Fourier series (period $= 2\pi$)
- $S_N(x) = \sum_{-N}^{N} c_n e^{inx}$. Notice that $S_{2N-1} = S_{2N}$.
- Find the sum $\frac{d S_{2N-1}}{dx}$. Let x_{2N-1} denote the first maximum point (to the right of 0) of $S_{2N-1}(x)$.



- Show that

$$\lim_{N\to\infty} S_{2N-1}(x_{2N-1}) > 1$$

② Let

$$f(x) = \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)(1+(x-t)^2)}.$$

Find $\widehat{f(x)}$. Use the result to evaluate the integral above. Hint: $e^{-|x|}$.

(3) Define

$$T(\phi) = \lim_{\varepsilon \to 0+} \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} \frac{\phi(x)\,dx}{x^2+\varepsilon^2}$$

when $\phi \in C_0^\infty(\mathbb{R})$. Show that $T(\phi) = \phi(0)$, i.e., Dirac's delta δ appears.

(4) Let $h(x) = x$, $-\pi < x \le \pi$, and $h(x+2\pi) \equiv h(x)$. Solve the equation

$$\sin(3x) = \int_{-\pi}^{\pi} h(x-y)\,g(y)\,dy, \qquad g(y) = ?$$

(5) Find the Fourier transform of the distribution

$$\langle Y, \phi \rangle = \int_{-\infty}^{\infty} \frac{\phi(x)\,dx}{|x|^{1/2}}, \qquad \phi \in \mathcal{S}(\mathbb{R}).$$

<u>Hint:</u> Use

$$f_N(x) = \begin{cases} |x|^{-1/2}, & |x| < N, \\ 0, & |x| > N. \end{cases}$$

<u>Fresnel's integral:</u>

$$C(s) = \frac{1}{\sqrt{2\pi}} \int_0^s |x|^{-1/2} \cos(x)\,dx$$

$0 < C(s) < 1$, when $s > 0$. $C(\infty-) = \frac{1}{2}$.