FOURIER ANALYSIS

① $$(1+q)(1+q^2)(1+q^4)\cdots = \frac{1}{1-q}, \quad |q|<1.$$

② Prove $$\frac{\sin\theta}{\theta} = \prod_{n=1}^{\infty} \cos\left(\frac{\theta}{2^n}\right)$$

and deduce Viète's formula (1593)

$$\frac{2}{\pi} = \sqrt{\tfrac{1}{2}}\cdot\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}\cdot\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}}\cdots$$

③ Let f be continuous in $[a,b]$. Find

$$\lim_{n\to\infty}\int_a^b f(x)\cos^2(nx+n^3)\,dx.$$

④ Verify $$\int_{-\infty}^{\infty}\frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{ab(a+b)}$$

where $a>0$, $b>0$. <u>Hint:</u> $\widehat{e^{-a|x|}} = \cdots$

⑤ Suppose that $f \in C^N(\mathbb{R})$ has period 2π. Prove that

$$\left|\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}\,dx\right| \le \frac{\text{Const}}{|k|^N} \qquad (k=\pm1,\pm2,\dots)$$

(6) If $f \in C(\mathbb{R})$ has period 1, then

$$\lim_{N \to \infty} \frac{f(\gamma) + f(2\gamma) + \cdots + f(N\gamma)}{N} = \int_0^1 f(x)\,dx \quad \text{(H. WEYL)}$$

when γ is irrational. Exhibit a function f such that the above formula fails with $\gamma = \frac{355}{113}$.

(7) Are the following functionals distributions in $\mathcal{D}'(\mathbb{R})$?

a) $T(\phi) = |\phi(0)|$

b) $T(\phi) = 1$

c) $T(\phi) = \sum_{n=0}^{\infty} \phi^{(n)}(0)$

d) $T(\phi) = \int_{-\infty}^{\infty} |x|^2 \phi(x)\,dx$

e) $T(\phi) = 2 + \int_{-\infty}^{\infty} \phi(x)\,dx$