4/4a, 4/4b, 4/5 (Chapter 4)

(1) Let $0 < s < 1$ and

$$u(x) = -\frac{1}{2} \int_{\mathbb{R}^n} \frac{\phi(x+y) + \phi(x-y) - 2\phi(x)}{|y|^{n+2s}} \, dy$$

for $\phi \in \mathcal{S}(\mathbb{R}^n)$. Show that

$$\hat{u}(\xi) = C_{n,s} |\xi|^{2s} \hat{\phi}(\xi)$$

In other words, except possibly for the constant $C_{n,s}$,

$$u(x) = (-\Delta)^s \phi(x)$$

<u>Remark</u> At least the case $n = 1$ is straightforward. You do not have to evaluate the constant $C_{1,s}$.

(2) Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1 - \cos(2\pi \langle y, \xi \rangle)}{|y|^{3+2s}} \, dy_1 \, dy_2 \, dy_3 \qquad (0 < s < 1)$$

$$= C_s |\xi|^{2s}.$$

First, notice that the integral is absolutely convergent. Then verify that it depends only on $|\xi|$, "not on ξ". Change coordinates.

SOME HELP

$$\xi = (\xi_1, \xi_2, \dots, \xi_n), \quad x = (x_1, x_2, \dots, x_n)$$

$$\langle x, \xi \rangle = x_1 \xi_1 + x_2 \xi_2 + \dots + x_n \xi_n$$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x) e^{-2\pi i \langle x, \xi \rangle} \underbrace{dx_1 \cdots dx_n}_{dx} \qquad \text{DEF.}$$

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{+2\pi i \langle x, \xi \rangle} d\xi \qquad \text{INVERSE}$$

$$\widehat{\frac{\partial f}{\partial x_k}} = 2\pi i \xi_k \hat{f}(\xi) \qquad \text{DIFFERENTIALS}$$

$$\widehat{f(x+a)} = e^{2\pi i \langle a, \xi \rangle} \hat{f}(\xi) \qquad \text{SHIFT}$$

$$\| f \|_2 = \| \hat{f} \|_2 \qquad \langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle \qquad \text{PLAN-CHEREL}$$

$$\widehat{f * g} = \hat{f} \cdot \hat{g} \qquad \text{CONVOLUTION}$$

$$\widehat{\Delta f(x)} = -4\pi^2 |\xi|^2 \hat{f}(\xi) \qquad \text{LAPLACE}$$

$$\widehat{(-\Delta)^s f} = (2\pi |\xi|)^{2s} \hat{f}(\xi) \qquad (0 < s < 1)$$

Def. of Fractional Laplacian.