① Let $G(x,t) = 1 + 2\sum_{n=1}^{\infty} e^{-n^2 t} \cos(nx)$, $t > 0$.

Jacobi's θ_3

Show that

$$G(x,t) > 0 \quad (t>0,\ -\infty < x < \infty).$$

<u>Hint</u> Verify that

$$G(x,t) = \sqrt{\frac{\pi}{t}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{(x-2n\pi)^2}{4t}\right)$$

(<u>Remark</u>: $u(x,t) = \frac{1}{2\pi}\int_0^{2\pi} G(x-y,t)\, f(y)\, dy$

solves $u_t = u_{xx}$, $u(x,0) = f(x)$, where $f(x+2\pi) \equiv f(x)$, period 2π.)

② Give an example of a rapidly decaying function $\phi \in C^\infty(\mathbb{R})$ such that its derivatives are not rapidly decaying. (By assumption

$$\max_x |x^n \phi(x)| < \infty, \quad n = 0, 1, 2, \dots \;)$$

③ Prove Wirtinger's Inequality

$$\int_0^{2\pi} |f(x)|^2 dx \le 1 \cdot \int_0^{2\pi} |f'(x)|^2 dx$$

Is 1 the best constant?

where $f, f' \in L^2(0, 2\pi)$, $\int_0^{2\pi} f(x)\, dx = 0$.

④ $\widehat{\sin(x)} = ?$ (in the sense of distributions)

(5) Assume for simplicity that $f \in C_0^1(\mathbb{R}^2)$. Assume that

$$\int_{\ell} f(x,y)\, ds = 0$$

along every line ℓ. Prove that $f \equiv 0$.

<u>Hint:</u> Use

$$\hat{f}(\xi,\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i(\xi x + \eta y)} f(x,y)\, dx\, dy.$$

<u>Notice</u> A priori, f is sign changing.

(6) Def.
$$\begin{cases} (\mathcal{F}\phi)(\omega) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int e^{-ix\omega}\phi(x)\, dx = \hat{\phi}(\omega) \\ (\overline{\mathcal{F}}\phi)(\omega) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int e^{+ix\omega}\phi(x)\, dx \quad \left(= \check{\phi}(\omega)\right) \end{cases}$$

$$\begin{cases} \hat{T}(\phi) = (\mathcal{F}T)(\phi) = T(\mathcal{F}\phi) & (\text{Def.}) \\ (\overline{\mathcal{F}}T)(\phi) = T(\overline{\mathcal{F}}\phi) & (\text{Def}) \end{cases}$$

Verify

$$\overline{\mathcal{F}}\mathcal{F}T = \underset{\substack{\uparrow \\ \text{Identity}}}{I} = \mathcal{F}\overline{\mathcal{F}}T,$$

i.e.

$$\mathcal{F}^{-1} = \overline{\mathcal{F}} \qquad (\text{Inverse})$$