(2) Absolute convergence:

$$0 \le 1 - \cos(2\pi\langle \bar{y}, \bar{\xi}\rangle) = 2\sin^2(\pi\langle \bar{y}, \bar{\xi}\rangle)$$

$$\le 2\pi^2 |\langle \bar{y}, \bar{\xi}\rangle|^2 \le 2\pi^2 |\bar{y}|^2 |\bar{\xi}|^2$$

$$\iiint_{\mathbb{R}^3} \frac{1 - \cos(2\pi\langle \bar{y}, \bar{\xi}\rangle)}{|\bar{y}|^{3+2s}} \, d\bar{y} = \iiint_{|\bar{y}| \le 1} + \iiint_{|\bar{y}| \ge 1}$$

$$\le 2\pi^2 |\bar{\xi}|^2 \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \frac{r^2 \cdot r^2}{r^{3+2s}} \sin\theta \, dr \, d\theta \, d\phi$$

$$+ \iiint_{|y| \ge 1} \frac{1 \, d\bar{y}}{|\bar{y}|^{3+2s}} = 2\pi^2 (4\pi) |\bar{\xi}|^2 \frac{1}{2-2s} + \frac{4\pi}{2s}$$

$$< \infty \quad \text{since} \quad 0 < s < 1.$$

$$f(\bar{\xi}) = \iiint \frac{1 - \cos(2\pi\langle \bar{y}, \bar{\xi}\rangle)}{|y|^{3+2s}} \, d\bar{y}$$

In $\mathbb{R}^3$,
$|\bar{\xi}|\bar{y} = \bar{z}$
$|\xi|^3 d\bar{y} = d\bar{z}$.

$$= \iiint \frac{1 - \cos\left(2\pi\left\langle |\xi| y, \frac{\bar{\xi}}{|\bar{\xi}|}\right\rangle\right)}{|\bar{y}|^{3+2s}} \, d\bar{y}$$

$$= |\bar{\xi}|^{2s} \iiint \frac{1 - \cos\left(2\pi\left\langle \bar{z}, \frac{\bar{\xi}}{|\bar{\xi}|}\right\rangle\right)}{|\bar{z}|^{3+2s}} \, d\bar{z}$$

when $\bar{\xi} \ne \bar{0}$, and $f(\bar{0}) = 0$. The integral is invariant under a rotation of $\bar{\xi}$:

$$f(A\bar{\xi}) = f(\bar{\xi}), \quad AA^T = I = A^T A$$
$$\det(A) = 1$$

Indeed,

$$f(A\bar{\xi}) = |A\bar{\xi}|^{2s} \iiint \frac{1-\cos\left(2\pi\langle \bar{z}, \frac{A\bar{\xi}}{|\bar{\xi}|}\rangle\right)}{|\bar{z}|^{3+2s}}\, d\bar{z}$$

$$= |\bar{\xi}|^{2s} \iiint \frac{1-\cos\left(2\pi\langle A^{-1}\bar{z}, \frac{\bar{\xi}}{|\bar{\xi}|}\rangle\right)}{|\bar{z}|^{3+2s}}\, d\bar{z}$$

$\bar{y} = A^{-1}\bar{z}$

$$= |\bar{\xi}|^{2s} \iiint \frac{1-\cos\left(2\pi\langle \bar{y}, \frac{\bar{\xi}}{|\bar{\xi}|}\rangle\right)}{|y|^{3+2s}}\, dy$$

$$= f(\bar{\xi})$$

Hence we may bring $\bar{\xi}$ to the position $(|\xi|, 0, 0)$ and write

$$f(\bar{\xi}) = |\bar{\xi}|^{2s} \underbrace{\iiint \frac{1-\cos(2\pi y_1)}{|y|^{3+2s}}\, dy_1\, dy_2\, dy_3}_{= C_s}$$

as desired. C_s is independent of $\bar{\xi}$.

①

$$\hat{u}(\xi) = -\frac{1}{2}\int_{\mathbb{R}^n} e^{-2\pi i\langle \bar{x}, \bar{\xi}\rangle} \int_{\mathbb{R}^n} \frac{\phi(\bar{x}+\bar{y}) + \phi(\bar{x}-\bar{y}) - 2\phi(\bar{x})}{|\bar{y}|^{n+2s}}\, d\bar{y}\, d\bar{x}$$

$$= -\frac{1}{2}\int_{\mathbb{R}^n} \frac{d\bar{y}}{|\bar{y}|^{n+2s}} \int_{\mathbb{R}^n} e^{-2\pi i\langle \bar{x}, \bar{\xi}\rangle}\left(\phi(\bar{x}+\bar{y}) + \phi(\bar{x}-\bar{y}) - 2\phi(\bar{x})\right) d\bar{x}$$

$$= -\frac{1}{2}\int \frac{e^{2\pi i\langle \bar{y},\bar{\xi}\rangle} + e^{-2\pi i\langle \bar{y},\bar{\xi}\rangle} - 2}{|\bar{y}|^{n+2s}} \hat{\phi}(\xi)\, d\bar{y}$$

$$= \int \frac{1-\cos 2\pi\langle \bar{y},\bar{\xi}\rangle}{|\bar{y}|^{n+2s}}\, d\bar{y} \cdot \hat{\phi}(\bar{\xi})$$

$$= C_{n,s}\,|\xi|^{2s}\,\hat{\phi}(\bar{\xi})$$

where example ② was used, strictly speaking done only in dimension $n = 3$.

<u>Remarks</u>: The double integral converges even absolutely, since

$$|\phi(x+y) + \phi(x-y) - 2\phi(x)| \le C|y|^2$$

say, when $|y| \le 1$, $x \in \mathbb{R}^n$. When $|y| \ge 1$ there is no problem.

$$\frac{d\bar{y}}{|\bar{y}|^{n+2s}} = \frac{r^{n-1}}{r^{n+2s}}\, dr\, dS_1$$

Fubini's theorem is applicable.