26.IV

FOURIER ANALYSIS

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Peter Lindqvist

① Show that

$$\sum_{k=-N}^{N}\left(1-\frac{|k|}{N+1}\right)e^{ikx} \geq 0.$$

② $$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy$$

$$\underbrace{f * f * \cdots * f}_{n \text{ factors}} = ?\,, \quad f(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

What happens as $n \to \infty$?

③ Let $\phi \in C_0^\infty(\mathbb{R})$. Show that $\hat{\phi}$ cannot have a compact support, unless $\phi \equiv 0$.

④ Assume that we know that [HEISENBERG]

$$\frac{\int_{-\infty}^{\infty}(x-a)^2|f(x)|^2\,dx}{\int_{-\infty}^{\infty}|f(x)|^2\,dx} \cdot \frac{\int_{-\infty}^{\infty}(\omega-\alpha)^2|\hat{f}(\omega)|^2\,d\omega}{\int_{-\infty}^{\infty}|\hat{f}(\omega)|^2\,d\omega} \geq \frac{1}{4}$$

for all $f \in L^2(\mathbb{R})$, when $a = 0$, $\alpha = 0$. Use this to prove the inequality for general a, α (real).

⑤ ~~6.~~ Assume that the functon f is in the Schwartz class $\mathcal{S}(\mathbb{R})$ and that it is "band-limited":

$$\hat{f}(\omega) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega x} f(x)\,dx \quad = \quad 0 \quad \text{when} \quad |\omega| > \pi.$$

Establish the formula

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx \quad = \quad \sum_{-\infty}^{\infty} |f(n)|^2.$$